

Second stage of Israeli students' competition, 2011.

Duration: 4 hours

1. In each vertex of a connected simple graph a number is written. The following action is repeated infinitely many times: all numbers are replaced simultaneously by the average of their neighbors. Consider the sequence of numbers which appear at a specific vertex of the graph. Assume that one of those sequences does not converge. Prove that the graph is bipartite (which means that its vertices can be painted in black and white so that neighbors are always of opposite colors).

2. Is it possible to find a planar strictly convex equilateral pentagon, all vertices of which are in \mathbb{Z}^3 (integer three-dimensional points)?

Remark. A polygon is called equilateral if all its sides are of the same length. It is possible for a polygon to be equilateral but not regular.

3. There is an urn with 5 balls: 2 blue and 3 white. Every minute, a random ball is chosen from the urn and returned with another ball of the same color. What is the limit of the probability that less than a half of the balls are blue, as the time goes to infinity?

4. We have a hyperbola and two distinct points A and B on it. For any additional point X on the same hyperbola, we define 3 numbers:

α = the distance from X to the straight line which is tangent to the hyperbola at A.

β = the distance from X to the straight line which is tangent to the hyperbola at B.

γ = the distance from X to the straight line AB

Prove that $\frac{\alpha\beta}{\gamma^2}$ doesn't depend on the choice of X.

5. Compute $\int_0^1 \frac{\ln(1+x)}{1+x^2} dx$.

Good luck!