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RESPONSE TO THE PAPER "WHAT SHOULD BE THE CONTEXT OF AN ADEQUATE SPECIALIST UNDERGRADUATE EDUCATION IN MATHEMATICS?", BY RONNIE BROWN AND TIM PORTER

DAVID WELLS

Brief Introduction

I have believed for as long as I can remember that understanding of any mathematical topic should include, and would benefit from, and appreciation of context—it's history, where it comes from, the original problems which prompted its creation, it's links to other topics and so on—so when I spotted the word 'context' in this article by Tim Porter and Ronnie Brown [**2**], whose work on maths education I have always very much appreciated, my attention was grabbed and I was prompted to respond.

My comments

All quotes from Ronnie Brown and Tim Porter are from [**2**].

> **Brown & Porter**: The main focus of arguments on undergraduate degrees in mathematics is on *content*. However an old school debating society tag is: "Text without context is merely pretext." We argue here that much of the implication of this remark holds in mathematics teaching.
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> The "context" of the training consists not only of the place it is given, and the background of the students, but also: the relation of the course to the rest of mathematics; what constitutes "good mathematics"; the possible future employment of the students; the way in which mathematics is used in society; the intrinsic value of the subject; how it

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> has progressed over the ages; and so on.
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> The point we are making is that the study of aspects of this context should be a clear part of the training. This reflects the fact that in any human activity, we need to know the background. If we decide to go on holiday, we don't rush to the station to buy a ticket: we consider what kind of holiday we want; what we can afford; what we can cope with; how to travel; and so on.

David G. Wells: I entirely agree with this linking of content to context. It occurs to me that if my course at Cambridge have presented any context at all—in addition to the standard content—then I might well passed my degree instead of going up on a scholarship and then failing. While I was at school I read all the books on the history of mathematics, of which there were several, in the school library, and I have always wanted—or even needed—to know the history and background of what I am trying to learn, and to be able to place it in context, as you put it.

It is very noticeable that some of the students I tutor do have a strong need to understand the meaning behind ideas while others are quite happy, apparently, to simply learn the technique. To what extent those students who want to understand the meaning would benefit further by appreciating the broader context I have no means of knowing, because I never have the time to explore such aspects which are peripheral to the examinations they are being prepared for.

I use the term *appreciation* because that is how I think of this theme. Some of my earliest articles were on the subject of appreciation and mathematics, and the importance of students understanding the context of what they were learning.

It is notable that the Cockcroft report did referred to appreciation at one point. Paragraph 240 commented:

> "In the teaching of mathematics it is possible to distinguish between three elements: facts and skills, conceptual structures, and general strategies and appreciation."

My own written submission to the Cockcroft Committee as a member of the great British public was largely about appreciation and its importance.

Unfortunately, the focus of commentators and the subsequent emphasis by mathematics educators was largely on paragraph 243 which asserted that mathematics at all levels should include problem-solving applications to everyday situations and "investigational work".

The idea of "general strategies and appreciation" seemed to slip out of general consciousness and was never developed further.

> **B&P**: Another important point is: what is the impression of mathematics that graduates gain from their studies?

DGW: Agreed. The same question applies to training mathematics teachers, most of whose studies include nothing about the history of mathematics or context in any other sense.

It is notable, for example, but although mathematics is used in all the sciences and especially in physics and chemistry, there are no references to the sciences at all in typical secondary modern mathematics textbooks (there used to be occasional references, decades ago).

Incredibly, it is quite common to see equations or formula introduced, which are quite clearly taken from elementary mechanics or optics, but this reference is entirely omitted so that, as far as the student is concerned, the terms of the equation or formula might have been invented at random.

(Currently, the other side of that coin, unfortunately, is that almost all mathematics has been removed from GCSE physics and chemistry.)

> **B&P**: How many professional Mathematicians use Galois theory, how many Pure Mathematicians use Fluid Mechanics? How many Mathematics Teachers use either? The reader may object—but what about Calculus, Linear Algebra, Rigid Body Mechanics, etc., the *really basic* material. Of course, there probably is a 'core' of material without which the student will be unable to understand what is going on in the subject at the present time, and so unable to operate as a specialist Mathematician. But it is not our purpose here to argue for the inclusion or exclusion of certain items in such a 'core'. Instead, we wish to ask: what is the core "for"? why is it the core? how is it to interact with other material? by what criteria should we gauge if something should, or should not, be in the core, and to some extent, how should core material be taught and assessed? These questions cannot be answered sensibly without considering the *context* of the mathematics course.

DGW: Following on from the last point, I prefer to think of General Concepts which are about mathematics—emphasis on about—and therefore are no part of conventional content although they do help to orient the student, to provide motivation for individual topics and themes and to create large numbers of links all of which not only aid learning and memory but also help to make sense of a very large amount of material which might otherwise seem formless.

> **B&P**: Twenty years ago, a document prepared for the London Mathematical Society, the Royal Statistical Society and the Institute of Mathematics and its Applications [**4**], states that a report on reducing student overload in first degree courses in engineering in the UK, included a suggestion to "Teach only the mathematics ... applicable to their chosen kind of engineering degree" and a proposal to "Reduce analytical theory".

DGW: As you are no doubt aware, complaints that the secondary syllabus for GCSE and before that for the O-level examination, go back decades. Cockcroft noted that the syllabus was overcrowded and more recently Adrian Smith agreed, and suggested the statistics might be removed and given to another subject such as biology: needless to say, absolutely nothing happened.

> **B&P**: One aspect of such work is that these mathematicians will need to "know how to communicate" with others, who will usually not be specialist mathematicians. This implies that they must to some extent explain, and "teach" the Mathematics, but they must also be able to get into the mindset of their "clients" to see what mathematics might be available, or need to be developed or adapted, to express the ideas the engineers or biologists are using. Part of the professionalism, therefore, of any mathematician should be the ability to *communicate* Mathematics and so it seems reasonable to suggest that training in such communication should be part of the *education* of a mathematician, even though the majority will not go into the teaching profession. In fact, our experience showed that enabling this aspect of their professionalism, actually helps their technical expertise in other aspects of their degree course including of the 'core'.

DGW: This should also apply to teachers at every level, including primary. When small children are using balances for weighing, for example, and slightly older children are discovering that 2 kg at 50 cm balances 1 kg at 100 cm, connections can and should be made to the use of such ideas in everyday life, and science and technology.

I have done such experiments with nine in 10-year-olds and with the 15–16 year olds, and they are invariably impressed and intrigued. Unfortunately, the usual textbooks completely avoid such links between the most elementary mathematics and very elementary science (mechanics) and of course their teachers are very rarely trained to introduce such links into their lessons.

> **B&P**: No recent survey of employers' views has been made within the UK, the latest reasonably thorough one would seem to be the McClone Report of 1974, [**3**]. As summarised there, the employers' view of the strengths of the mathematics graduate include:
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> (i) knowledge of mathematical technique;
> (ii) ingenuity;
> (iii) capacity to seek out further knowledge;
> (iv) ability in problem solution.
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> On the negative side, the mathematics graduate is:
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> (i) poor at formulating problems,
> (ii) poor at planning work,
> iii) poor at making a critical evaluation of completed work, and in any case it did not matter too much what they did since
> (iv) they had little or no idea of how to communicate it to others.
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> Has much changed since 1974?

DGW: If anything, I suspect that the situation has got worse, in company with the general dumbing down of textbooks and examinations and syllabuses. An example: the textbook *General Science: Physics* by Spencer White published in 1936 and still being reprinted in 1952, written for O-level students, is about 350 pages long and mathematical formulae or data or graphs, or whatever, occur on about half of the pages, a frequency that is unthinkable in a modern textbook for the same age group.

A feature of GCSE examinations which has been the subject of frequent comment and criticism in recent years, is the manner in which the student is guided step-by-step, like a pig with a ring through its nose, through each examination question, in contrast to 50 years ago when the examinee might be faced with a bare question with no subquestions and no guidance as to how the two problem should be solved.

> **B&P**: Are there any subject specific skills that should overlay these general skills? It is easy to suggest a few:
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> (j) Understanding of the use of mathematics in the modelling of aspects of the "real world". (Design, analysis and limitation of models).
> (k) Appreciation of the conceptual and descriptive power of mathematics.
> (l) Appreciation of the notion of mathematical validity, that is to read, understand and write proofs.
> (m) Skills in avoiding 'slips' in calculation, etc.

DGW: Proof has been the subject of an explosive growth of interest among the secondary mathematics educators some of which seems to me to be more to do with empire building than with any genuine interest or understanding of proofs vis-a-vis secondary (or indeed primary) pupils.

In the December issue of the European Mathematical Society Newsletter there was a strange article by the EMS Education Committee (a footnote suggested that they all approved of the article without having all contributed equally to its composition) in which it was claimed among other things that the concept of proof is alien to students and must be treated as something entirely novel—a claim which is totally false and mistaken—and dangerously misleading.

I replied with a letter which is published in the most recent issue of the Newsletter, in which I referred, of course, to abstract games, puzzles, mathematical recreations and so on, in which many and varied contexts, ideas of proof (meaning a completely convincing argument where the question "convincing to whom?" is left hanging in the air) arise naturally and pose no problems at all too typical pupils provided they are introduced in the context of problems that they can solve and argue about, either with a teacher or among themselves.

A very simple current example—but by no means trivial in this context—is provided by the Sudoku puzzles in the newspapers, in which the solver, typically, can prove by watertight reasoning which every solver appreciates that a certain number must go into a certain cell.

The original extraordinary letter by the EMS education committee is a perfect example in my opinion of how academics who are not in touch with actual children and perhaps never were, can make a mountain out of a molehill.

(The same phenomenon also it sometimes seems accompanied by elements of empire building, can be observed in academic books and papers on the notorious theme of word problems and the difficulties that students have with them, at every age). Word problems are genuinely difficult because they require interpretation but they are not as difficult as some academics would like to make out.

> **B&P**: Further, Mathematics is presumed to be fully formulated, tidy and neat, not necessitating any leap of the imagination; rigour is the main characteristic, not intuition, or ideas; it is a difficult subject, with a high risk of failure. A global view of the purpose of the subject

is not available, and popularisation is almost impossible, or so it is implied.

DGW: the current renewed emphasis on problem solving in primary and secondary mathematics classrooms ought to do something to overcome this dry and thin picture of mathematics. It ought to, but may not. The NCTM in the USA announced that the 1980s would be the decade of problem-solving: the effect was significant, but limited.

B&P: To take another metaphor, in training a *chef*, one does not present the trainee *chef* just with finished meals, nor just with the task of peeling the potatoes. A trained *chef* is required to design and produce the finished product, the meals, to a high standard. In [**1**], the point is made that a valuable course in carpentry is one in which the student uses particular skills to make a finished product, on which the assessment could be based.

DGW: One of the problems of teaching mathematics is that it is more or less invisible [**6**, p. 19]. Several metaphors can be used to illuminate the nature of mathematical activity even when it cannot be observed directly. The simplest and most striking of these metaphors, for younger pupils at least, is that of the detective. Stepping back and taking an overall view of mathematical problem-solving including the aspects of motivation, persistence and determination, an excellent metaphor is the rock climber or mountaineer who deliberately selects challenges that are neither too easy nor too difficult but chooses those within reach, according to his or her judgement.

B&P: Solving problems was one of the mathematics graduates "strengths" in McClone [**3**]. Can we use it to help with the other skills?

Schoenfeld [**5**] described a problem solving course he ran. The aims of this course included helping the students to develop their mathematical judgment, to "understand, justify and communicate mathematical ideas". (We might add that by using various different styles of structure, Schoenfeld's course also allows students to work in teams. He did this in part by allowing the students to learn by making a tactical withdrawal from the class.)

He also tried to get students to see Mathematics as a human activity with a set of criteria for validity—but *not one* imposed from outside, *not that* decided only by the lecturer. The students evaluated their own efforts by defending their views and attacking contrary ones, until consensus as to the valid argument was reached. To do this, Schoenfeld created an artificial environment to provide students with "a genuine experience of *real* mathematics". He concludes:

> "By that standard, standard mathematics instruction is wholly artificial."

DGW: Alain Bouvier (and other teachers and maths educators) have experimented with slightly similar formats. He would present an "open" problem (*probléme ouverte*) to a classroom of secondary students, who would then split up into small teams to try and solve it. After a suitable time lapse, the teams would get together as a class and one member of each team would be appointed to explain the group's solution to the problem. It was then up to members of the other teams to critique this solution.

B&P: A technique widely used by psychologists and trainers is *error-less learning*. This falls into two types. One is where large hints, props, and supports to a specific course of action are given, and the action is rewarded as a symbol of success. Then the various props are gradually withdrawn. The other type uses *reverse chaining*: the easiest way to see to this is to think of encouraging a child to put on a vest. You do not throw him or her a vest and say put it on; instead, you put it almost on, and then ask the child to do the final action. Subsequently, you gradually put the vest less and less fully on, till the whole action can be done.

One way of using the last technique in university mathematics is to write out a formal proof and then erase bits of it. The student has to fill in the bits, using clues from the rest of the proof. This has some analogies with the practice of a professional mathematician, who may have an idea and outline for a proof, but needs to work on details. The student also gets an idea of the structure of a proof. Such an exercise is also very easy to mark!

The general feeling about error-less learning is that it works like a dream!

DGW: What a brilliant idea! It sounds like exercises in English (cloze tests) in which gaps in a paragraph or story has to be filled in by the student—but the mathematics version sounds more challenging and more interesting (but I may be biased).

B&P: In either method, the fact long verified by psychologists is used, that *we learn from success*. We can also learn to accommodate failure if that is gradually introduced, and strategies are available for dealing with failure.

DGW: See the comment above on the rock climber or mountaineer analogy and the student who is allowed—encouraged—required—to choose a problem which they themselves will tackle, on the understanding that if they decide after a while that it is either too easy or too difficult then they are allowed to drop the problem and choose another.

An important general concept here is that it is extremely difficult, very often, to decide how difficult a problem is (in the context of your current mathematical strength, as it were) before you have started to tackle it. Examples: Andrew Wiles taking seven years to prove Fermat's last Theorem—versus—several of Hilbert's 23 problems which were solved unexpectedly quickly.

B&P: In the UK, mechanics has disappeared from the options offered by many schools, being replaced by probability and statistics. As a school subject, Physics has also been on the decline for various reasons. Students arrive at university having had little opportunity to reflect on the interaction of mathematics with "mechanical" reality. As a result, students tend to find mechanics hard and even unintuitive or too abstract.

DGW: Yes, this is very sad—and not least because probability and statistics, important though they are in themselves, consists at school level largely of students plugging data into formulae or algorithms which they don't even begin to understand in terms of the genesis and—yes—context.

B&P: This has led several institutions in the U.K. to experiment with practical modelling sessions, in part reintroducing the "physics laboratory" session that would previously have been available in schools. The equipment needed has been developed jointly by a group of institutions and is designed to be relatively cheap to buy, mostly being assembled from "toys".

The idea of a laboratory session is old, but that does not mean it is "bad". It can provide opportunities for applying knowledge to the formulation of a model, problems (at what angle of slope, will the car fall off the circular track?), evaluation of results so far (my model predicted that at this angle the car would loop the loop—it didn't manage it; where was my model inadequate? how can I improve it?),

> working in teams and communicating the results in a report. Emphasis may be placed on only some of these skills but again the student is *doing* mathematics.

DGW: Very interesting. When I taught in a private primary school I was also teaching science, with a large amount of simple equipment summer which came from one of the parents who was a very successful builder with a large and fascinating yard. Naturally, the maths benefited from the science and conversely.

Roundabout a century ago, when the sciences such as physics had only recently been accepted as suitable subjects for secondary education, there were mathematics laboratories in several schools—I seem to recall that Oundle was one of them—and it was even suggested by one enthusiast that physics and mathematics should be taught together. What actually happened was that they rapidly drifted apart.

> **B&P**: Similarly, in operational research, several universities have experimented with bringing in *real* problems for the student(s) to handle. This can be as an individual project or as a group project on traffic flow, how a local timber firm can best cut or stack its wood to minimise waste or wasted space. The problem may be small, but real. This also provides opportunities for communication, since if a local firm has provided the problem, it should receive a readable report. Here, "readable" means "readable to the non-mathematical management of the firm". Group work is useful here both logistically, realistically and for the advantage to the students.

DGW: There are further ironies here. During the modern mathematics movement, linear programming appeared in many syllabuses, being itself a relatively novel and interesting development in the relatively new field of operations research. In order to solve very simple linear programming problems, pupils had to draw graphs of inequalities, and select the appropriate solution region.

50 or 60 years later, what are the results? The original linear programming problems disappear completely, linear programming does not feature in any current secondary syllabuses or examinations—BUT—the one feature which can be examined most easily, ENTIRELY OUT OF CONTEXT (!) is the textbook exercise of drawing very simple inequalities on a graph.

It is now an exercise not a problem, the real-world context has entirely vanished, and the connection with business has vanished, leaving only the dry husk of formalism. Unfortunately, this seems to be the natural tendency whenever anything novel and exciting and progressive and interesting is introduced!

> **B&P**: A course on Mathematics and Society at Liverpool University resulted from discussions with one of us on our course, and in that course also, the organisers, Roger Bowers and Brian Denton, say they were bowled over by the quality of the presentations. Thus it may be that the standard mathematics courses have failed to capitalise on the universal nature of mathematics, the fact we all need to work with the geometric, logical, numerical aspects of the world around us.

DGW: Generally speaking, I think it is true to say that whenever students are given the opportunity to be more responsible and more imaginative and more creative, they surprise their teachers—whether they are young pupils in primary school or somewhat older pupils in lower secondary school, or sixth formers—and no doubt, from what you are saying, university students.

I would say that this is one of the main messages of progressive education, that teachers consistently and persistently underestimate the abilities of pupils and students. They teach uninspiring content in uninspiring styles, and then when the students don't do as well as they are supposed to, the conclusion is drawn that the students have failed—whereas the correct conclusion more often than not is that it is the teachers who have failed.

It is a sad fact that so many teachers and maths educators never having taught in a progressive style or manner, are incapable of realising what's typical pupils are capable of achieving.

> **B&P** A third theme was the interaction of Mathematics with other Sciences and with Industry.

DGW: See comments above, including the comment on linear programming. I do believe that from the perspective of the pure mathematician and the applied mathematician using mathematics in science and industry, the typical school syllabus and textbook and examination, and therefore maths in most classrooms, represents no more than the dry husk of the corn, the crucial kernel having been threshed out—instead of threshing out the husks and keeping the kernels.

> **B&P**: For a couple of years it gave a small introduction to some Euclidean Geometry. There were three reasons for this.
> (i) This subject is part of our heritage, and students should surely know at least why the angle in a semicircle is a right angle.
> (ii) There are some lovely and surprising results.
> (iii) It introduces students to the idea of proof, where the proof proves something striking.

DGW: From the comments made above you will appreciate that I find the third point here strange, and dispiriting. It is frankly bizarre—though so common it seems as to be near-universal—that any student could get to university without having been introduced to ideas of proof and learning to appreciate the difference between a convincing proof and an unconvincing argument, many years before.

This is, again, especially relevant to the links between abstract games and mathematics—as in the letter mentioned earlier, published in the EMS Newsletter. [**7**]

> **B&P**: Finally, one difficulty mentioned by a visitor from Spain was that you can't teach context to the students until you have taught it to the staff. Brian Griffiths wrote similarly in an email to RB that lecturers have never had lectures of this type. All one can say in response is that it is sad if a cycle of deprivation continues.

DGW: I was very sorry when BG died. We occasionally corresponded—always with very interesting results. I fear that it would take a massive effort on the part of many people to overcome this vicious circle, especially given (1) the immense and continuing pressure to remove the kernel of every mathematical topic leaving only a formal husk, and (2) the organisation of the examination system including strong commercial pressures to continually dumb down, which have been exerted for decades and only recently recognised by politicians who have suddenly woken up to a phenomenon they should have spotted years ago—and to which their reaction is already turning out to be entirely inadequate.

To mention just one point, as crushing as it is obvious: for more than 50 years now the emphasis in school mathematics has been ostensibly on pupils learning to understand each topic, yet a level and both GCSE and A-level examinations have never ever been designed to test understanding. Since teachers feel under professional and moral obligations to help their pupils pass examinations, they inevitably have a very limited motivation for emphasising understanding—especially given that teaching for understanding takes longer than teaching for rote learning, and the syllabuses are generally agreed to be overcrowded.

> **B&P**: Rather we would argue that specialist and nonspecialist, teacher and researcher, should be exposed to discussions on the nature, context, history, of mathematics, and that in such a course each can give valuable contributions. All of our students should obtain a clear impression of a sensible professional approach to the subject.
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> A course for professionals, and this includes both specialist and non specialist, teacher and researcher, should include not only technique, and knowledge, but also a sense of value, an idea of what is "good mathematics", why it may be called good mathematics, and what are the areas of debate in such a judgement. Without context and value, the course becomes dehumanised, and students can become confused and so demoralised.

DGW: Hear! Hear!

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About the author

More years ago than I care to recall, I went up to Cambridge University on a scholarship but failed CIA to eye with my tutors—I thought the methods of teaching mathematics were terrible—and so fail my degree, though I did develop my skill in playing chess.

Subsequently I trained as a mathematics teacher of the progressive persuasion, teaching pupils from 9 to 16, and later became puzzle editor of the magazine *Games & Puzzles*, and a professional game inventor, and then a full-time author, mostly of popular maths books.

I kept in touch with mathematics education by reading, writing articles for maths ed journals, and a secondary mathematics course based on problem-solving, and occasionally attending conferences. For the last six years I have also been tutoring pupils of all ages from 8 to 25.