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MULTIPLE CHOICE EXAMS IN UNDERGRADUATE MATHEMATICS

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1. *Introduction*

In addition to a rigorous practical test called the general flying test, candidates for a private pilot's licence have to pass written exams in subjects including meteorology, navigation, aircraft, and communications. These written exams are multiple choice, which seems appropriate. The trainee pilots are acquiring skills supported by background knowledge in breadth not depth, and this can be tested by asking them to choose the right option from a limited list under a time constraint. It is not necessary, of course, for pilots to understand the underlying theoretical concepts.

In contrast, students of mathematics are certainly expected to understand underlying theoretical concepts. To a certain extent, this understanding can also be tested using multiple choice exams. Clearly, mathematicians need skills too, of which one of the most important is the ability to perform calculations accurately. This can also be tested using multiple choice exams.

Given that no one method of assessment is good for all of the understanding and skills expected of a student, one should use a variety of different assessment methods in a degree programme, including things such as vivas, projects, and conventional written exams. There is no claim here that multiple choice exams can do everything!

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2. *An example*

Here is an example of a two-hour exam at the end of a first year course on matrices, vectors, and complex numbers. It had the following rubric.

> *For full marks, candidates should attempt all questions. Any correct answer scores 3 marks, and any incorrect answer scores* -1 *mark.*

(1) If $i = \sqrt{-1}$ and

$$M = \begin{pmatrix} w+z & x+iy \\ x-iy & w-z \end{pmatrix},$$

which of the following is $\det(M)$?

[**A**] $w^2 + x^2 + y^2 + z^2$ [**B**] 0
[**C**] $w^2 - x^2 - y^2 - z^2$ [**D**] $w^2 - x^2 - y^2 + z^2$

(2) If $M = \begin{pmatrix} 7 & 4 & 1 \\ 2 & 0 & 1 \\ 3 & 2 & 0 \end{pmatrix}$, which of the following could be M^{-1}?

[**A**] $\begin{pmatrix} -2 & 3 & * \\ 2 & * & * \\ * & * & * \end{pmatrix}$ [**B**] $\begin{pmatrix} -1 & 1 & * \\ 3/2 & * & * \\ * & * & * \end{pmatrix}$

[**C**] $\begin{pmatrix} 2 & -5 & * \\ 1 & * & * \\ * & * & * \end{pmatrix}$ [**D**] $\begin{pmatrix} -1 & -2 & * \\ 5 & * & * \\ * & * & * \end{pmatrix}$

(3) If $a \neq 0$, how many solutions do these equations have?

$$x - y + 3z = a,$$
$$2x + y - 5z = 2a,$$
$$x - 4y + 14z = 3a.$$

[**A**] none [**B**] exactly one
[**C**] more than one, but a finite number [**D**] infinitely many

(4) If

$$M = \begin{pmatrix} 1 & 3 & 2 \\ 0 & 2 & 4 \\ 3 & 0 & -4 \end{pmatrix},$$

which of the following is the largest of its eigenvalues?

[A] -1 **[B]** 3 **[C]** 4 **[D]** 7

(5) If the eigenvectors of the matrix M in the previous question are written in the form (x, y, z), how many of them have $x = 0$?

[A]	none	**[B]**	exactly one
[C]	exactly two	**[D]**	all of them

(6) $OABC$ is a parallelogram, with $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OC} = \mathbf{c}$. The point P is the mid point of OC, and the point Q is four fifths of the way along $\overrightarrow{AP}$. OQ intersects BC at R. Which of the following is the ratio $BR : RC$?

[A] $1:1$ **[B]** $1:2$ **[C]** $1:3$ **[D]** $2:3$

(7) Let

$$\overrightarrow{OA} = (3, 1, 2)$$
$$\overrightarrow{OB} = (4, 0, 7)$$
$$\overrightarrow{OC} = (2, 1, 1)$$
$$\overrightarrow{OD} = (0, 2, -5)$$

If the lines AB and CD intersect, denote the angle between them by θ. Which of the following statements about these two lines is correct?

[A] they intersect and are perpendicular
[B] they intersect and

$$\cos\theta = \frac{11}{\sqrt{3}\sqrt{41}}$$

[**C**] they do not intersect
[**D**] they intersect and

$$\cos\theta = \frac{33}{5\sqrt{39}}$$

(8) Consider

$$\mathbf{a}.(\alpha\mathbf{a} + \mathbf{b}) \times (\lambda\mathbf{a} + \mu\mathbf{b} + \mathbf{c}).$$

To which of the following is it equal?

[**A**]	$\mathbf{a} \cdot \mathbf{b} \times \mathbf{c}$	[**B**]	0
[**C**]	$\alpha(\lambda + \mu)\mathbf{a} \cdot \mathbf{b} \times \mathbf{c}$	[**D**]	$\mathbf{0}$

(9)

$$S = \{(7, 0, 2, 0), (0, 3, 1, 1), (1, 2, 0, 0), (5, -1, 3, 1)\}.$$

Here are three statements about the set S of vectors.

(i) S spans $\mathbb{R}^4$.
(ii) S is linearly independent.
(iii) S is a basis for $\mathbb{R}^4$.

[**A**]	none is true	[**B**]	exactly one is true
[**C**]	exactly two are true	[**D**]	all are true

(10) Let V and W be two subspaces of a vector space U, neither V nor W being a subset of the other. Consider their intersection $V \cap W$ and their union $V \cup W$ as subsets of U. Which of the following statements is correct?

[**A**] neither $V \cap W$ nor $V \cup W$ is a subspace
[**B**] only $V \cap W$ is a subspace
[**C**] only $V \cup W$ is a subspace
[**D**] both $V \cap W$ and $V \cup W$ are subspaces

(11) Division by $1+i\sqrt{3}$ has the following effect on the complex number z

[**A**] it moves down by 1 and to the left by $\sqrt{3}$
[**B**] it rotates anticlockwise through $\pi/2$ and its length is increased by a factor of 2
[**C**] it rotates clockwise through $\pi/2$ and its length is unchanged
[**D**] it rotates clockwise through $\pi/3$ and its length is decreased by a factor of 2

(12) If the modulus of $z-1$ is equal to the real part of $z+1$, then z must

[**A**]	be zero	[**B**]	lie on a certain straight line
[**C**]	lie on a certain ellipse	[**D**]	lie on a certain parabola

(13)

$$\arg(\cos\theta\cos\phi - \sin\theta\sin\phi + i(\sin\theta\cos\phi + \cos\theta\sin\phi))$$

is equal to

[**A**] 0 [**B**] $\dfrac{\sin(\theta+\phi)}{\cos(\theta-\phi)}$ [**C**] $\tan(\theta+\phi)$ [**D**] $\theta+\phi$

(14) Here are three statements about the roots of $z^8 = 1$.

(i) They add to zero.
(ii) All but one of them come in complex conjugate pairs.
(iii) They all lie on a circle centre the origin.

[**A**]	none is true	[**B**]	exactly one is true
[**C**]	exactly two are true	[**D**]	all are true

(15) $(1/4)\text{cosec}^2(\theta/2)$ can be expressed as

[**A**] $\dfrac{1/2}{e^{i\theta}+e^{-i\theta}}$ [**B**] $\dfrac{e^{i\theta}}{e^{2i\theta}+1}$

[**C**] $\dfrac{e^{i\theta}}{e^{i\theta}-e^{-i\theta}}$ [**D**] $\dfrac{-e^{i\theta}}{(e^{i\theta}-1)^2}$

3. *Some comments on the questions*

A short cut in question 1 would be to put $x = y = 0$, which this examiner would regard as fair game. In question 2, the candidate only has to calculate two of the components of M^{-1}, but this is enough to test whether the entire inverse could in principle have been calculated. The asterisks prevent "reverse engineering". Instead of the obfuscating option [C] in question 3, one could put "The information given does not determine a specific answer". Questions 4 and 5 are phrased carefully to prevent reverse engineering: the candidate has no choice but to calculate the eigenvalues and eigenvectors. In question 6 we could also have used the option "The information given does not determine a specific answer". The structure of question 9, asking how many statements are true, is a very flexible way of dealing with non mutually exclusive properties.

4. *Discussion*

As is evident, the course emphasised *technique*, on the grounds that a deeper understanding depends upon the student being able to perform these elementary calculations reliably. However, there is an even starker point.

Surely we require our mathematics graduates to be able to calculate things correctly, just as we require our pilots to be able to take off and land safely. But in most assessment regimes it is quite possible get a mathematics degree without ever having got a significant calculation right. This is a good reason to include multiple choice exams, especially in the first year: they are a clear signal to the students that accuracy is important.

What about understanding, though? Even this more elusive quality is being tested in the above exam, especially in questions 10 and 14, but also, perhaps to a lesser extent, in other questions. Merely the successful parsing of question 6, for example, requires some understanding.

People who loathe questionnaires—and the author is one—are naturally suspicious of multiple choice exams. But designed with care, this form of assessment can be subtle and valuable.

About the author

After graduating in mathematics from London University in 1975 Stephen Huggett studied for his DPhil in Oxford under the supervision of Roger Penrose. He also learned to fly, gaining his private pilot's licence in 1979, before he learned to drive. He still has a deep interest in twistor theory, but his current research activity is in polynomial invariants of graphs and knots. He was Programme Secretary of the LMS from 2001 to 2011, and is Secretary of the European Mathematical Society.

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