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A DRAFT SCHOOL MATHEMATICS CURRICULUM FOR ALL WRITTEN FROM A HUMANE MATHEMATICAL PERSPECTIVE: KEY STAGES 1–4

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Abstract

Note: This draft was hammered out by a small group, which included experienced school teachers, textbook authors, curriculum administrators, and mathematicians. In particular, many helpful suggestions from Tony Barnard, Richard Browne, Rosemary Emanuel, and David Rayner have contributed to the current version. It offers a *mathematician's-eye-view* of school mathematics to age 16, which we hope will serve as a useful focus for wider discussion and debate.

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Contents

Background

The attempt to draft a comprehensive set of goals for school mathematics, which is coherent, rigorous, and imaginative constitutes a daunting challenge. How can one best convey the *essence* of elementary mathematics? How can one do this in a way that serves (almost) *all* pupils, *all* teachers and *all* schools? And how can one structure it so as to provide a useful guide, rather than a straitjacket? In presenting this draft, we recognise the difficulty of combining these goals within a single document.

Despite this difficulty, most educational systems try to provide such a statement—even if many acknowledge these conflicting goals by specifying a *common* curriculum only up to some point (such as the end of Key Stage 3). The resulting scheme can then

– serve as a communal objective (e.g. so that no-one who might complete the full programme is deprived of the opportunity to do so), and
– provide a map, a guide, or a framework for teachers and pupils, for educational administrators, and for parents and employers.

The act of devising a curriculum is inevitably a top-down process, in which the drafters select and interpret certain higher objectives. Our declared mathematical focus means that we have concentrated on delineating the elementary *mathematical* material which, if mastered more-or-less in the given sequence, might prepare pupils better for life and—where appropriate—for subsequent study.

How the document should be structured to avoid misunderstandings is a different—and a more delicate—question. The current situation in England is confused. International comparisons raise serious questions about our approach to 'differentiation' in school mathematics, and suggest that it has effectively institutionalised underachievement. The *Strategies* recognised the importance of 'whole class teaching', and showed how it can be used to bring on *the whole class*. At the same time, teaching and assessment actively encourage early decisions that set a ceiling on what is expected of large groups of pupils. The pressure to adopt very early 'setting', and to treat each pupil 'as an individual', has been accepted without demur—even though we know that such approaches often reinforce low expectations. (This point is forcefully made in the recent Ofsted triennial review of school mathematics teaching *Mathematics: made to measure* (May, 2012), p.65.) So there is at present no cosy consensus as to how a serious curriculum *for all* might be structured. We have tried to choose a uniform framework that specifies achievable goals for *all* students—including those for whom the complete programme is too ambitious at this stage; but given the lack of a local consensus, we hesitate to make inflated claims for the particular approach we have adopted.

The draft describes a curriculum of elementary mathematics for age 5–16. Taken as a whole, it specifies a basic curriculum for those (roughly the top 50–60%) who may choose to proceed to further academic study of some kind at age 16. At the same time we have tried to present this scheme in such a way that the content listed within Key Stages 1–3 could provide an appropriate foundation for *all* pupils as long as schools have the scope to interpret it flexibly. In other words:

– we see the material listed in Key Stages 1–3 as a curriculum goal **for all**—knowing that some pupils will take longer to lay this foundation (which implies that teachers should have sufficient flexibility in Years 9–11 to revisit and interpret the relevant listed content to suit their audience)

– we have also tried to ensure that the 'more abstract' material which is then listed under Key Stage **4** can build on this shared platform in such a way that Key Stages **1–4** constitute a coherent but minimal coverage of the elementary mathematics that is needed as a foundation at age 16 *for those who wish to proceed to further academic studies* (whether in numerate disciplines or not).

Evidence from the UK and elsewhere suggests that—in due course—we could expect a significant majority of pupils to complete the full programme by age 16. But in the first instance, some will proceed more slowly; some will need to revisit key themes after the age of 16; and some, having absorbed as much as they can from this classroom-based approach, may choose to develop their use of mathematics in other ways.

One possible misunderstanding became clear only after we thought we had completed our work. We were confronted by criticisms which indicated that the individual statements in our draft were being interpreted as if they were fragmented "outcomes", which were to be taught and assessed as *separate items.*

In contrast, we had interpreted the challenge of devising a "curriculum" as an attempt to delineate a **journey**, or educational experience, with a clear mathematical destination. Hence the material specified here should not be seen as a mere bureaucratic list of *outcomes*, but as a sequence of *stages*, whose combined function is to prepare the way for subsequent mathematics, providing a platform which should leave pupils well-placed to develop subsequent key ideas and methods.

Despite the apparent danger that some might misconstrue a curriculum presented as a structured list of 'key technical details', we reject the fashionable alternative of imagining that one can focus instead on a small number of "big ideas". In a society where mathematics teachers have a shared mathematical culture it may be possible to suppress mention of details to some extent, because they are so well-known that they would not then be lost; but that is not the situation in which we find ourselves. In particular, whilst there is much truth in the maxim that

> "education is what remains when every fact you have learned has been forgotten",

this should **not** be interpreted as saying either that the details (and their sequencing) are unimportant, or that one can specify a worthwhile mathematical education in terms of some collection of "big ideas". For while the intended *thrust* of the curriculum is more important than any individual statement, this core thrust can only be conveyed (implicitly) through the choice of the individual steps, and the way they are worded and linked together. If one tries to bypass the central task of teasing out this network of details, the result is inevitably weak and unconvincing (as with the 2007 *National Curriculum* for England).

Thus it would be wrong to parody the relatively tight focus of the programme given here as being somehow 'retrograde'. We have actively tried to learn from the experience of the last 60 or so years. In this spirit we have avoided taking a rigid stance with regard to certain developments which are bound to impinge on the mathematics *classroom*, but which we do not see as being part of the *mathematics* that is to be taught. For example, we should all understand both the potential advantages and the pitfalls of adjuncts to the mathematics classroom such as calculators, computers, investigations, and certain applications of mathematics. The claims made on their behalf may have often been too 'optimistic', but these tools and approaches are there to be used—even if it remains unclear *how* they should best be used to support 'what the *beginner* needs to master'.

The focus here on core ideas is driven by a serious desire to devise a modern *curriculum-for-all*, which leaves scope for teachers to make use of any approaches and tools which have proved their worth, or which they find useful. In particular, teachers need the freedom to choose how to adjust the way they approach basic material for different groups. But insofar as elementary mathematics is to be taught to *all* pupils, *what* is taught (and in what sequence) has to respect and to reflect both the inner structure of elementary mathematics and the way human beings learn. For example, if we know that access to mathematics at higher levels requires mastery of fractions, or of algebra, our task must be to help as many students as possible to achieve these goals. This then determines much of the material that *all* pupils need to master if they possibly can. And schools should not decide too quickly who is, and who is not, capable of such mastery. We must not accept a situation in which a substantial number of pupils are offered a diluted curriculum and substandard teaching from an early age (as shown in the Ofsted report *Mathematics: made to measure*, May 2012), so guaranteeing that they will never progress to higher levels. This does **not** mean that we should encumber the *National Curriculum* with the language of 'entitlement' (which leads to the current pretence that all 14–16 year olds are being given the same diet), but that teachers should understand what material holds the key to subsequent progress, and should be given the flexibility to help as many pupils as possible to achieve an effective grasp of these core techniques.

Whatever framework is adopted, we insist that elementary mathematics should never be conceived as a single *ladder* up which different pupils scamper as fast as they can. (This is a crude description of the TGAT model that underpinned the original 1989 *National Curriculum*, and which has plagued us ever since.)

Additional constraints arose from what we judged to be key local realities. In particular, whenever a choice had to be made between

(i) moving on to more sophisticated methods before the end of a given Key Stage, and

(ii) postponing this move, whilst using the time to develop a stronger, more concrete, platform on which *the next* Key Stage might build,

we tended to err on the side of the latter—if only because we judged that we were then more likely to succeed in training teachers at each Key Stage to teach the specified material successfully.

Each of our curriculum statements looks both *backwards* and *forwards*:

- it looks *backwards* in that it depends on, and develops, earlier material, and
- it looks *forwards* by constructing a platform on which pupils can base their subsequent learning.

There is a *minimal* interpretation of each curriculum statement which is consistent with these two perspectives. But each curriculum statement can also be interpreted more richly. Pupils need to master each chunk of material as robustly as they can. In particular, at each stage, *more able* pupils need to master the core material *far more thoroughly and flexibly* than is specified here for the average pupil—precisely because what they learn will subsequently need to sustain a more extensive mathematical 'superstructure'. So for those who may subsequently proceed to further studies, what is presented here needs to be supplemented and strengthened (rather than accelerated!), while for those who proceed more slowly, what is listed here will need a degree of interpretation as well as selective pruning. Notwithstanding these remarks, *we have tried to organise the material so that—up to a point—the underlying sequence and progression makes sense even if covered more slowly.*

The observant reader will notice a total silence on the matter of *levels*. This is a natural consequence of the recognition that each statement or idea can be mastered to differing *depths*, and our consequent rejection of the interpretation of the statements as if they were rungs in some shared 'ladder' up which each pupil climbs at his or her preferred rate. Each pupil's degree of mastery of each technique grows with time—as it is used. At a given stage a teacher only needs to know whether the depth of mastery already attained is sufficient to sustain the next steps—but the job is never done, and familiar techniques and ideas need to be constantly reinforced and revised. The whole thrust of the 'curriculum journey' we have tried to outline here is compromised if teachers are obliged to think in terms of continually assigning 'levels' (i.e. to certify when mastery of each technique has finally been 'achieved'). Weighing the pig never helped the pig to grow; but at least it keeps track of something real (the pig's *weight*); in contrast, the way pupils come to master mathematical ideas cannot be captured by the educationally destructive practice of ascribing an imagined *level* to each pupil in each curriculum domain.

Successive administrations have made the mistake of thinking that a *mathematics* curriculum should be relatively easy to specify. One can understand the mistake: the content of elementary mathematics has been around for many centuries, so the necessary debates about inclusion and sequencing have a long history. But the central position of elementary mathematics as a school discipline means that it is subjected to political and social pressures which often muddy the waters. The best mathematics teachers care deeply about their subject; but for too long they have been subjected to unhelpful swings between

– *freedom* (sometimes a cover for official inability to decide which of 'a range of possible methods' to recommend),

and

– *central control* (sometimes overtly threatening, sometimes intended as guidance), which too often seeks to help teachers by over-specifying details in a way that obscures the essential core.

Both extremes leave teachers, schools (and pupils!) **at the mercy of the assessment system**. When the mathematics curriculum is poorly, or too tightly, specified, what is seen as important no longer depends on mathematics, or on the way pupils learn, but on how schools feel they must 'play the system' in order to achieve certain grades on examinations. That is, too often what passes for school mathematics in the classroom is driven by *bureaucratic* pressures, rather than being guided by the subject itself and by teachers' professional judgement. (This point is forcefully made in the 2012 Ofsted report *Mathematics: made to measure*.)

If *by the age of 16* the vast majority of pupils are to achieve a significant mastery of most of the material that is listed here *up to the end of Key Stage 3*, then we need a framework which gives schools and teachers freedom to introduce key ideas in a suitable way. Schools and teachers must have the flexibility to pace material in a way that matches the needs of those who struggle, the freedom to regularly revisit and reinterpret material from earlier phases, and the authority to bypass certain topics where they judge it appropriate to do so. In the same spirit, pupils who may later proceed to further numerate studies should be expected to master elementary mathematics in greater depth than is outlined here. Modulo such adjustments, the core ideas listed here *up to the end of Key Stage 3* are meant

– to constitute a mathematical platform for those who then proceed to Key Stage 4 and to further studies beyond,

and at the same time

– to serve as a basic toolkit for those whose mathematics may ultimately serve more mundane ends.

The draft has been gestating since the mid-1990s, when the *London Mathematical Society* (with the IMA and RSS) published the report *Tackling the mathematics problem*, and subsequently responded in detail to the consultation in connection with the 1999 *National Curriculum*. It seeks to provide a coherent outline of those aspects of elementary mathematics

– which constitute a central part of our intellectual heritage, and
– which provide a foundation for further study and for adult life.

The goal is thus *universal*; but since we are operating in a UK context, we have had to make choices. In particular, we have adopted a *format* loosely based on previous versions of the local (English) *National Curriculum.*

Many observers with a mathematical perspective have been encouraged recently by an apparent convergence in the wider mathematics education community that recognises the need for the school curriculum to respect the *inner structure* of elementary mathematics: in particular, we have been impressed by sources as varied as evidence from the *Strategies*, from HMI/Ofsted, from TIMSS, from colleagues who study mathematics teaching in European countries, from the process of devising the *Core Curriculum Standards* in the USA, and from developments in the Far East. So, while the origins of this particular specification lie primarily within the UK academic community (including teachers and others who value a mathematical perspective), the resulting specification is neither blindly elitist nor parochial.

The recent White Paper *The importance of teaching* (2011, p. 40) declared that

> *The National Curriculum should set out only the essential knowledge and understanding that all children should acquire, and leave teachers to decide how to teach this most effectively.*

In one sense, this is exactly what a *National Curriculum* should be. Yet this principle can be interpreted too narrowly. There is much to be said for leaving teachers freedom to innovate and to adapt to local circumstances. But this admirable aspiration must not be allowed to give rise to premature dogmatism as to what can safely be excluded from the curriculum. In particular, what needs to be specified *at a given time*

– depends in part on what is, and what is not, second nature to those who are likely to consult the document *at the time it is written.*

This draft incorporates a clear position as regards the balance which needs to be struck in a shared, centrally devised mathematics curriculum between

> the *What?*, and the *How?*.

The distinction between what counts as *essential knowledge* and what constitutes *how to teach* is not as clear cut as the italicised aspiration from the White Paper suggests. The developmental sequence and the logical hierarchy of elementary mathematics do not always fit comfortably together during the early years; but, the *links* ***between*** *mathematical ideas* are a key part of "the essence of elementary mathematics"; hence specifying a particular sub-goal (part of the *What?*) has consequences for the intended route to that goal, and for the links that need to be established with other sub-goals (the *How?*). Unfortunately, such matters are generally neglected, and their importance would seem not to be widely understood. This neglect of the interactions between sub-goals becomes more telling as one proceeds to higher levels, and this is reflected in the amount of detail we have had to provide in the *Fuller version* of this draft curriculum for each Key Stage. This additional detail is provided in part to clarify what is intended—and so to facilitate discussion and debate; but it is also spelled out because the need to develop elementary mathematics systematically, and the kind of consequences this should have for instruction, are not sufficiently appreciated.

Thus our approach embraces the spirit of the White Paper's aspiration. But we have softened its impact in three ways.

First, (following the example of the 1999 Curriculum) some material, which has been suppressed in the main text on the grounds that it goes beyond that which should be explicitly specified, has been included in the form of *examples* [prefixed by "e.g." in square brackets]. These examples are intended for clarification, and are not strictly part of the curriculum itself: the exact details could be modified, but they are still meant to be taken seriously.

Second, we have been less strict about policing the border between the *What?* and the *How?* in the *Fuller version* of the draft curriculum for each Key Stage. The *Brief version* may therefore be closer to the prototype of a 'minimal statutory document', whilst the *Fuller version* may be seen as a kind of *non-statutory guidance*, which could be regularly improved and updated. For the most experienced teachers, the statements in the *Brief version* may be sufficiently self-explanatory; but others may need to consult the *Fuller version* in which the intended meaning of certain curriculum statements is elaborated more explicitly.

Third, we have allowed ourselves this introductory preamble, or *Background*, where the goals and status of the project are outlined and some of the features of the final version are explained informally. In settings where there is a common culture among mathematics teachers, such things can often be encapsulated relatively briefly. As an example consider the following short comments in the *Explanatory note* which goes with the Russian primary curriculum (for ages 6–10):

> *Mathematics teaching in primary school should build pupils' mathematical concepts and skills in such a way as to ensure successful mastery of mathematics in secondary school. Pupils learn four arithmetic operations, master algorithms for oral and written calculations, learn to evaluate numerical expressions, and solve word problems. Children form spatial and geometric concepts. All programme material is presented spirally, so that abilities and skills are steadily deepened so as to develop pupils' conscious reflection on mathematical activity.*
> *The mathematical content should be used to facilitate general educational capabilities, skills and types of activity, and to develop links with other subjects.*
> *A primary school pupil will get an idea of natural numbers and zero; will understand numeration and the decimal system; will learn how to perform oral and written arithmetic operations with numbers (up to a million); will learn to find an unknown in an orally presented arithmetical operation [*without using algebra*]; will learn the meaning of the relations "more (or less) than [*a given quantity*]", and "[*so many*] times more (or less) than';, will be happy with the rules for the order of operations; will get an idea of quantities (or magnitudes); will work with standard geometric figures; and will learn to solve simple word problems.*

In a setting where there is a shared mathematical culture, such a brief summary may well suffice. But in the current English context we appear to be converging rather slowly toward a shared mathematical culture, so it is not easy to keep this *Background* succinct. In particular, we include below a short elaboration of twenty one groups of **key terms** which appear in **boldface** in the curriculum statements, but which we were advised needed explicit clarification.

There are many possible formats for a *National Curriculum.* Some are definitely more helpful than others. But gratuitous novelty in design can make it harder to convey the intended message. We judged there to be a significant consensus that, in contrast to previous versions and to the 2007 version, the 1999 *National Curriculum* constituted a reasonable model to use as a first approximation. So rather than invent an entirely new framework (which would still be to some extent arbitrary), we have tried to work within the general framework of the 1999 Curriculum, adapting the overall structure and organisation where there was good reason to do so. In particular:

- to avoid the temptation to specify content more precisely than is appropriate, we embrace the 1999 notion of *Key Stages* (which break up the Years 1-11 into four 'stages': $11 = 2 + 4 + 3 + 2$) as a suitable way of grouping material and of delineating intermediate endpoints
- each Key Stage begins with an indication of *Breadth of Study*, which summarises selected broader goals to give an overview of the intended character of the Key Stage
- this is followed by a detailed listing of the *Knowledge, Skills, and Understanding* to be taught—grouped at each stage under just *two* main headings; the chosen headings evolve as one moves from Key Stage 1 to Key Stage 4, but they consistently reflect the basic division of elementary mathematics into two distinct categories—the *numerical/symbolic* and the *geometrical/spatial*
- each main heading is then further subdivided, with the subheadings (and sub-subheadings) evolving to reflect the change in character of the material as one moves through the Key Stages; these subheadings constitute convenient pegs on which to hang the intended *content*, *sequence*, and *connections*, and allow one to indicate briefly the approach which is needed if the intended goals are to be achieved.

There was some debate in 1999 about simplifying and clarifying the main headings for the *attainment targets.* This debate was continued in the 2004 Smith report *Making mathematics count.* Here we have adjusted the names used, and the distribution of material in line with these debates, in ways which appear to sharpen the intended message.

- Thus at Key Stage 1 and Key Stage 2, *Number and measures* seeks to emphasise the links between *number* and *quantity*, and the way a child's experience of number at this stage is embedded in counting and measuring. At the same time, *Shape, space, and measures* (which becomes *Geometry and measures* at Key Stage 2) emphasises a similar link. (In neither case do we see the need to respect some imagined bureaucratic edict that *measures* should only arise under one heading.)
- What is naturally presented informally as *Shape, space, and measures* at Key Stage 1 evolves naturally towards *Geometry and measures* at Key Stage 2, and simply *Geometry* at Key Stages 3 and 4, to reflect the shift of attention which occurs when one identi-

fies, and begins to explore and to analyse individual shapes, and to organise the study of geometry from a *mathematical* viewpoint.

– Given the limited mathematical tools that can be developed by the average pupil up to age 16, we have not felt able within this draft mathematics curriculum to resolve the important question of how to achieve a degree of universal "statistical literacy"—but we suspect a satisfactory answer will require a perspective that transcends any given subject. In line with the implied recommendations of the 2004 report *Making mathematics count*, we have adjusted the position of *Handling data*. In previous versions it has been listed as a separate attainment target, which effectively reduces the time available for more basic material (and is often given even more classroom time because of the way it is assessed). This draft interprets *Handling data* as an important sub-theme of *Number and measures*, or *Number and algebra* (and graphical representation). This should not be interpreted as an attempt to downplay the importance of the material, but rather (as emphasised in *Making mathematics count*) to stress that

> the effective mathematical analysis of "variability" *depends on* achieving a *prior* level of fluency in number, in algebra, in graphical work, and in geometry.

The last 20 years have witnessed all sorts of attempts to ensure that the elementary mathematics taught in school can be *used*—ranging from including a separate curriculum strand called *Using and applying mathematics*, to the quicksands of obligatory 'coursework', and the spurious attempt to invent a new subject called 'functional mathematics'. As these examples indicate, we still need to sort out what our objectives should be on this front, and how we might best achieve them. In the light of these historical difficulties, this draft adopts a simple strategy. We have shifted the *Breadth of study* statement to the very beginning of each Key Stage, in order to emphasise the intended spirit; and in the spirit of trying to focus more coherently on 'essential content', we have chosen to absorb *Using and applying* into each of the two main headings—as part of the intended curriculum rather than as an attempt to tell teachers *how to teach*. In this spirit we have retained the subheadings

– *Problem solving*
– *Communicating*
– *Reasoning*

as a reminder

– that whatever content' is covered needs to involve pupils in *working on problems*, and to bestow an increased confidence that the material they have learned allows them to tackle mildly unfamiliar problems more effectively (though the heading is deprived of its hyphen, to avoid giving the impression that there is some magic hyphenated art called 'problem-solving')
– that the process of learning, and learning to use, elementary mathematics depends on talking, reading, writing, presenting solutions, and refining one's language in a way that makes meanings crystal clear
– that this process of refining ideas, and the way language is used, opens up the possibility of *objective proof*, or certainty, so that techniques and results learned, and the answers obtained, are uniquely reliable in a way that is part of the 'essence of elementary mathematics'.

The draft for each Key Stage started from scratch with a blank sheet. We first worked on the *Fuller version*, whose statements were then organised using a modification of the framework adopted for the 1999 Curriculum (with variations of the headings and subheadings). This draft was then extended, pruned and revised after consulting various pre-existing curriculum documents from the UK and elsewhere, and by incorporating comments and criticisms.

Once the *Fuller version* was in reasonable shape for a given Key Stage, the shorter *Brief version* was extracted from it. Finally, and partly tongue-in-cheek, a *one page summary* of each Key Stage—the *Very Brief version*—was produced, which sought to capture something of the spirit of its goals, whilst losing most of the detail.

Each Key Stage focuses on the *What?* of elementary mathematics. This can be crudely summarised as:

– content
– sequencing, and
– key connections.

To some extent the second and third bullet points in the above summary are already in conflict with the aspiration expressed on page 40 of the White Paper, which could be taken to suggest

– that a centrally devised curriculum should be limited to a mere list of *content*, and
– that the way the content is *sequenced*, and the emphasis placed on *connections* between topics, are best left to the individual teacher or the school.

This view may contain a grain of truth; but it ignores essential features of elementary mathematics and the purpose of a *shared* curriculum.

In sequencing given content, some choices are a matter of judgement; but others are in some sense independent of any particular implementation (e.g. where the ordering is determined by the intrinsic logical character of the material, or by what we know about human development). In such cases, the *sequence* becomes part of the intended structure implied by the listed *content*. Similarly, the numerous *connections* between separately listed ideas and methods impinge directly on the development of the material, yet are widely neglected in many English schools (which as the Ofsted report *Mathematics: made to measure* (May 2012) observed often "present mathematics as sets of disconnected facts and methods that pupils need to memorise and replicate"). The explicit mention of these *connections* is therefore essential if the meaning of the listed content is to be grasped.

Sometimes the need to include features which appear to stray beyond the *What?* arises

– for forward-looking *administrative* reasons (for example, to ensure that teachers of a given cohort in subsequent years have a known foundation on which to build), or
– for forward-looking *developmental* reasons (that is, to optimise pupils' inner preparedness for subsequent mathematics, where material at one stage has to be approached in a particular way if the experience is to motivate and provide an effective basis for later mathematical learning).

These general comments indicate some of the more obvious instances where the *What?* of 'essential knowledge and understanding' intrudes on the *How?* (i.e. *how* specified material is to be approached, or *how* a curriculum statement is to be interpreted). Other instances are less clear cut. Nevertheless, it has been our intention wherever possible that the *How?* should be left for schools and teachers to decide.

The reasons for the choices that have been made often remain implicit in the structure and flow of the draft. We need teachers who can appreciate, analyse, and criticise this implicit structure. Whatever curriculum is adopted, we require an *urgent, and systematic programme* to strengthen secondary teachers' subject knowledge and their subject pedagogical knowledge (as was plainly revealed in the 2008 Ofsted report *Mathematics: understanding the score*, and the recent sequel *Mathematics: made to measure* (May 2012)). We also need to recognise the damage done by persistent rhetoric about "driving up standards", and by central control that abuses the existing processes of accountability and assessment. Thus the process of turning the aspirations underlying a revised curriculum into an improved classroom reality will require a concerted effort, mainly from the State,

– to devise a suitably flexible *implementation framework*
– to redesign assessment and accountability procedures so that performance data are used to *support* positive moves for improvement;

and efforts from the State and profession working together

– to improve textbooks, subject knowledge, assessment, etc., and
– to provide the requisite continuing professional development.

We close this introductory section by providing an outline of the intended meaning and status of twenty-one themes — key words and phrases used in the draft, which we were told needed further elaboration. These themes are:

1. basic facts and techniques, connections
2. learning by heart, fluency, automaticity
3. exercises, problems
4. counting: the product rule
5. place value, compression
6. standard written algorithms
7. multi-step problems
8. word problems, *realistic* problems
9. ratio, proportion, the unitary method
10. inverse problems, simplification
11. structural arithmetic
12. divisibility tests
13. symbols: expressions, formulae, equations, identities
14. sequences: intrinsic definitions, closed formulae
15. exactness
16. estimates, approximation
17. Euclidean geometry: a *formal* approach
18. ruler and compass constructions
19. proof, line-by-line format
20. probability, populations, samples
21. calculators

1. Basic facts and techniques, connections

Elementary mathematics *should* be experienced as an ever-expanding, but increasingly unified whole. Topics that at first appear unrelated, or only loosely linked, regularly come to be seen as different instances of one and the same underlying theme. For example, fractions, ratio, percentage, etc. make their initial appearances in different chapters, but should sooner or later be recognised as being different embodiments of the same basic idea. As time passes, the total amount of mathematical material increases; but the **connections** between old and new ideas allow it to be reorganised, so that at each stage a relatively small number of **basic facts** hold the key to a remarkable range of techniques. These *connections* between topics, and the way they help to simplify the material, capture *the essence* of elementary mathematics far more profoundly than an (inevitably) fragmented list of topics.

In practice we realise that pupils often experience elementary mathematics as an ever-increasing list of unrelated, and even apparently arbitrary, rules! The whole thrust of this draft is to resist—even to reverse—this tendency.

- Euclid in 300 BC had no general way of representing numbers, and so had to approach measures through his elusively subtle 'theory of proportion'; in contrast, we can now represent integers using *place value* and the digits 0–9 (an idea due to the Hindus and Arabs around 1000 AD); we then extend this notation to decimals (thanks to Stevin in 1585 AD). This allows us to link *measures* with our ideas of *number* (starting from counting numbers, and extending first to fractions, then to negatives, surds, etc.).
- The ancient Babylonians (around 1700 BC) solved quadratic equations in a classical disguise, by giving rules for finding positive unknowns whenever their *sum and product* were given (though they treated only certain special cases and then found only one solution). In extending this ancient approach to cover all possible cases, Cardano (*Ars Magna* 1545 AD) was obliged to solve three different kinds of equations, because his coefficients always had to be positive. Hence $x^2 = 2x + 3$ had to be treated separately from $x^2+2 = 3x$, and $x^2+3x = 10$; and each type led to further subdivisions, since "square equals unknowns plus number" sometimes had a solution and sometimes not. More importantly, if all the variables and coefficients have to be positive, no quadratic equation could ever have the form $ax^2 + bx + c = 0$!

In contrast, by first mastering negative numbers and elementary algebra, secondary pupils can now 'complete the square' and solve all quadratics in a uniform way.

The inescapable constraints on the way a 'list' of required material is printed, and on the way it has to be read (in sequence), leads to the separation *on the page* of topics and ideas which should ultimately be linked—both *in the classroom* and *in pupils' minds.* Nevertheless, the listing given here does what it can to emphasise *connections*, and to convey a philosophy of routinely linking new ideas and methods back to a limited collection of *basic facts and techniques.*

Once one moves beyond elementary mathematics, one has to accept that the justification of proven methods in terms of *basic facts* or ideas often requires one to grasp *extended* deductive chains, or proofs. The draft curriculum here includes examples which illustrate this occasional need for longer chains of reasoning. However, the spirit permeating the draft is that school mathematics should always 'make sense'. If students are to eventually appreciate 'distant' logical dependence, they need an intensive apprenticeship during which the logical justifications for standard methods are easy to understand, and are regularly reinforced as each method is used. This need to emphasise how derived methods follow from *basic facts* is part of the *What?*, so needs to be explicitly specified. Once the basic repertoire of easily derived methods is securely grasped, students may be better placed to understand and use results whose derivation requires longer chains of reasoning.

2. Learning by heart, fluency, automaticity

In recent years those who decide what a typical pupil should be expected to learn and to remember have downplayed the importance of *memorisation.* Yet there are all sorts of reasons why we need to **learn** certain things **by heart**.

George Steiner has written eloquently about the way memory contributes to 'what we are'. Our concerns here are more prosaic. Pupils certainly need to be completely on top of that limited collection of basic facts and techniques in terms of which most elementary mathematics can be understood. But they need to memorise far more than this. For example, when tackling an unfamiliar problem, one must be able

– to consider a number of possible approaches and intermediate steps in order to assess the most promising strategy; and

– for this to be possible, the steps or techniques one has to weigh up as part of this process must all have been robustly internalised.

On a mundane level, when faced with routine *inverse* problems (see **10.** below)—such as "simplify $\frac{36}{54}$"; or "calculate $102^2 - 98^2$, or "factorise $x^4 - 7x^2 + 1$"—one cannot begin unless the relevant direct facts are known *by heart* so that we have a chance of recognising that they are needed (e.g. $36 = 4 \times 9$, $54 = 6 \times 9$; or $a^2 - b^2 = (a - b)(a + b)$, without which one is unlikely to notice that $102^2 - 98^2 = 4 \times 200$, or that $x^4 - 7x^2 + 1 = (x^2 + 1)^2 - 9x^2$). That is, pupils need to memorise enough to allow them to respond flexibly.

> *What you don't know by heart, and so can't access instantly, you can't use.*

This observation applies not only to facts (such as $36 = 4 \times 9$), but also to *procedures*: that is, we need to attain **fluency** in handling a wide range of arithmetical, algebraic, trigonometric, and geometrical procedures, so that each new procedure can eventually be exercised *automatically*, quickly, and accurately. Once this level of **automaticity** is achieved, the brain is left free to focus on those more demanding aspects of a problem that require genuine thought (like trying to express $x^4 - 7x^2 + 1$ as a difference of two squares).

3. Exercises and problems

An **exercise** is a task, or a collection of tasks, that provides *routine practice* in some technique, or combination of techniques. The techniques being exercised will have been explicitly taught; and the meaning of each task will be clear. All that is required of pupils is that they implement the procedure as taught to produce an answer. Each collection of such exercises should be designed to 'exercise' the given skill in order to establish mastery of the relevant technique in a suitably robust form. In particular, a well-designed set of exercises should highlight, and help to eliminate, standard misconceptions and errors.

Exercises are not meant to be particularly exciting, or especially stimulating; but they give pupils a quiet sense of satisfaction. And without a regular diet of suitable *exercises*, ranging from the simple to the suitably complex, pupils are likely to lack the repertoire of basic techniques they need in order to make sense of mildly more challenging tasks.

Exercises are like the bread-and-potatoes of the mathematics curriculum. But a healthy diet needs more than just bread and potatoes. Pupils need more challenging activities to help whet their mathematical appetites, and to cultivate an inner willingness to tackle, and to persist with, slightly unfamiliar tasks. A **problem** is any task which we do not immediately recognise as being of a familiar type, and for which we therefore know no standard solution method. Hence, when faced with a *problem*, we may at first have no idea how to begin.

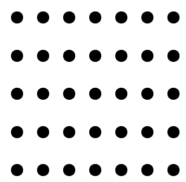
A task does not have to be very unfamiliar before it becomes a *problem* rather than an *exercise*! In the absence of an explicit problem solving culture, an *exercise* may appear to the pupil to be a *problem* simply because its solution method has not been mentioned for a week or so, or because it is worded in a way which fails to announce its connection with recent work.

Too many exercises get stuck at the level of 'one piece jigsaws', so conveying the message that mathematics consists exclusively of mindless repetition. Hence, at the very least, each set of *exercises* should include tasks that force pupils to think more flexibly, to string simple steps together in a reliable way, and to discover the astonishing increase in power that results. In short, pupils need to learn from their everyday experience that the whole purpose of achieving fluency in routine bread-and-potatoes *exercises* is for them to be able to marshal these skills to handle slightly more demanding multi-step *exercises*, and more interesting, if mildly unsettling, *problems.*

A more subtle question is whether *exercises* and *problems* should be carefully 'graded'—with the teacher always beginning with the *easiest* cases; or whether the main examples tackled with the whole class should be carefully chosen to provide students from the outset with *deeper insight*. How challenging one can safely be may well depend on the class. But experience from those who observe lessons in other countries suggests that our inclination to start with *easy* cases may ignore the extent to which this encourages pupils to depend on *ad hoc* methods that do not extend to the *general case*. So we need to consider the benefits of starting with *exercises* that bring out the full extent of the complexity we wish pupils to master, or with harder *problems* that oblige pupils to think flexibly.

4. Counting: the product rule

This draft curriculum stresses the way calculation changes its character as we move towards the end of primary school (moving from blind computation to an emphasis on exploiting algebraic structure through *structural arithmetic*—see **11.** below). In the same way, **counting** larger collections also demands a more structured approach, based on the **product rule**. In its simplest form, the *product rule* reduces to the observation that a '5 by 7' array of dots



contains 7 dots *in each of* its 5 rows, and hence contains

$$7 + 7 + 7 + 7 + 7 = 5 \times 7$$

dots altogether.

This innocent-looking observation provides a delightful way of counting the number of factors of any positive integer—such as 189. First factorise the given integer as a product of prime powers:

3) 189
3) 63
3) 21
      7

So $189 = 3^3 \times 7$. Now every factor of 189 must have the form $3^a \times 7^b$, where $a = 0$, 1, 2, or 3, and $b = 0$, or 1. To count the possible factors $3^a \times 7^b$ of $189 = 3^3 \times 7$, we count the possible pairs (a, b):

- there are 4 possibilities for a (the power of 3), and
- *for each power of* 3, there are 2 possibilities for b (the power of 7),
- so there are $\mathbf{4} \times \mathbf{2}$ possible factors altogether—including the factors 1 (when $a = b = 0$), and 189 (when $a = 3$, $b = 1$).

The same idea allows one to estimate the size of a large crowd. First we divide the crowd into a number n of 'blocks' of roughly equal size; then we count, or estimate, the number b of people in a given 'block'; finally we calculate $n \times b$ (see **16.** below).

5. Place value, compression

Mastering the art of working freely with number, measures and number operations by using the powerful notation of **place value** remains one of the central goals of early school mathematics. Our ability to talk about and to work with numbers changed dramatically with the introduction of *base* 10 notation. This notation is surprisingly recent: Fibonacci introduced the Hindu-Arabic notation for integers into Europe in the early 13th century, and Stevin proposed a cumbersome extension to decimals shortly before 1600 AD. Modern notation then evolved during the 17^{th} century.

The modern way of representing numbers by combining *position* and *powers of* 10 is the simplest example of a persistent theme in elementary mathematics, namely **compression**, whereby unimaginably elusive worlds are captured in simple form. *Place value* allows an ungraspable string of tally marks, such as

||||||||||||||||||||||||||||||||||||||||||||||||||||||||||||||||||||||||||

to be first grouped into 10s

|||||||||| |||||||||| |||||||||| |||||||||| |||||||||| |||||||||| |||||||||| ||||

and then represented simply as “74” (7 tens and 4 units). This combines two simple ideas:

- first use position, or *place* to represent *value* (‘units’, ‘tens’, ‘hundreds’, etc.),
- then notice that each *place* contains a digit between 0 and 9, and that counting and calculating can therefore often be reduced to working with the digits 0-9 (which everyone can manage).

6. Standard written algorithms

Our astonishingly *compressed* notation for numbers is further exploited when we come to mechanise the procedures for calculating. The ideas of addition and subtraction, of multiplication and division, are best grasped slowly, in various practical ways, and are exercised initially through regular mental work and informal written methods. But when dealing with larger numbers, we need robust *general calculation methods*—or algorithms; and by far the most effective calculation methods are the ‘column arithmetic’ algorithms for addition and subtraction and for long multiplication and short division—referred to here as **standard written algorithms**. These ‘standard column procedures’ for addition, for subtraction, for long multiplication, and for short division miraculously reduce ‘general calculation’ to *repeated implementation of operations with the digits* 0–9—i.e. to number bonds and tables. This compresses potentially complex procedures into manageably compact form by combining *place value* and a thorough knowledge of the basic addition and multiplication facts for digtis 0–9.

The algorithms for integer arithmetic later extend naturally to allow calculation with decimals—which underlines yet another reason why these algorithms are here built in as explicit *content*.

Long division also reduces harder problems to a standard format; but its intermediate steps *straddle a number of columns*. It therefore seems inappropriate for early introduction, and may be felt to be appropriate for only around half of the cohort as an excellent example of an algorithm (perhaps completed using a calculator to help find appropriate multiples of the dividend), and to help prepare for decimal division, and later polynomial division. {Long division—like a few other topics—is included in this draft curriculum in curly brackets, to indicate that it should be taught, but is not intended for *all* pupils.}

7. Multi-step problems

The draft makes repeated reference to the need to solve **multi-step** problems. A *multi-step* problem is like the challenge to cross a stream that is too wide to straddle with a single jump, so that the prospective solver is obliged to look for 'stepping-stones'—intermediate points which break up the otherwise inaccessible problem of crossing from one bank (what is given) to the other (the completed solution) into a chain of manageable steps. In elementary mathematics, this art has to be learned the hard way. It should not be seen as optional, or as a matter of taste; it is part of the *What?*, not of the *How?*.

In one sense, a chain consists of a set of separate links. But its usefulness (and indeed its essence) arises from the way its individual parts are *interlinked*. Without this interlinking, the chain is useless. In the same way, basic routines become useful only when they can be linked together to solve more interesting problems.

Efforts to improve the effectiveness of mathematics instruction in recent decades have sought to monitor centrally what is taught and learned by specifying 'outcomes'. Too often we have then cheated by deeming these 'outcomes' to have been achieved when the integrated technique that needs to be mastered has in fact been broken down into 'atomic parts', which are then taught as a sequence of *separate* curriculum statements, and assessed as a sequence of *single steps* (the *one piece jigsaws* referred to in **3.** above). We have lost sight of the fact that the *essence* of elementary mathematics lies in mastering each *integrated technique* 'as a whole'—not in mastering the individual steps alone, but in learning to see, and to exploit, the *connections* between them, and to coordinate these steps to solve simple *multi-step* problems.

A curriculum or syllabus can specify the individual techniques, or steps; but this is futile if one then forgets that it is the ***linking*** of the content which determines whether these techniques can be effectively used. This interlinking is an elusive property, *which depends entirely on the way the material is taught:* that is, it depends on the teacher. Central prescription, and political pressure to demonstrate relentless year-on-year improvement, have resulted in a didactical blindspot, with curriculum objectives and assessment—and hence teaching—becoming fragmented, so that pupils are only expected to handle 'one piece jigsaws'. Exams routinely break down each problem into a succession of easy steps—in order to minimise the risk of failure, and to ease 'follow-through marking'. Teachers then conclude that they can ignore the delicate art of *interlinking* simple steps; and we all pretend that candidates have achieved mastery of the relevant *integrated* technique. This is a delusion. The individual steps may be a starting point, but the power of elementary mathematics lies in *the way simple ideas can be combined* to solve problems that would otherwise be beyond us. That is, the essence of the discipline lies in the *interconnections.* Hence the curriculum and its assessment need to cultivate the ability to tackle *multi-step* problems without them being artificially broken down into steps.

8. Word problems, 'realistic' problems

Successive versions of the mathematics *National Curriculum* displayed an admirable determination to incorporate *Using and applying* within teaching and assessment. But the experience of the last 20 years is more useful as a guide to what does *not* work than to what does work. In particular, we have neglected simple **word problems**, which should be part of pupils' daily diet from the earliest years.

Word problems are problems which typically consist of two or three short sentences, where the task is given in words, leaving the pupil to extract the meaning and any required information, and then to identify what needs to be done. The simplest examples require pupils to identify, to extract, and to use information, and to choose and implement procedures in a way which is typically *multi-step.* The requirement for the pupil to read and to extract the relevant data from two or three English sentences constitutes the first (apparently routine—yet surprisingly elusive!) 'stepping-stone' *en route* to a solution.

During Key Stage 1 *word problems* are important because they reflect the fundamental links between

– the world of mathematical ideas and mathematical reasoning,

and

– the world of language, and the logical universe inherent in its grammar.

At later stages *word problems* continue to serve as an invaluable way of linking the increasingly abstract world of mathematics to the world where its ideas can be applied: that is, they can be seen as the simplest instances of any programme designed to ensure that elementary mathematics *can be used.*

But *word problems* are only a beginning. Pupils also need a regular diet of tasks which forge a stronger link between the world of mathematics and the power it offers pupils to tackle problems from the wider world. There are those who advocate using 'real-world' problems. These may look superficially appealing; but they are almost always unsuitable. In much the same way, the claim that calculators and computers allow pupils to work with 'real (= dirty) data' ignores the distracting effect of 'noise', which often compromises the attempt to use 'real' contexts. Problems which support the move towards 'using and applying' beyond the limited world of *word problems* need to be carefully constructed, so that the real context truly reflects the mathematical processes pupils are expected to use as part of their solution. This draft occasionally uses the word **realistic** to describe problems which, though artificial, have been carefully constructed to convey the spirit of *application*, have been crafted so as to generate the intended level of thought, and have been tested to ensure that they lead predictably to mathematical analysis of the intended kind, while avoiding the usual pitfalls.

Since Jim Callaghan's Ruskin speech of 1979 we have witnessed a consistent concern about pupils' ability to *use* the elementary mathematics they are supposed to know. Much effort has been expended in trying to do better—but with limited effect. In particular, ambitious attempts to coerce change—using extended investigations, coursework, and modelling—have mostly served to demonstrate what should **not** be officially required at this level. Our response is both more and less ambitious than what has been attempted in previous proposals. It is less ambitious in that the explicit encouragement is for *more focused* tasks; it is more ambitious in that we advocate a *permanent thread* of such focused material from the earliest years. (We hope that individual teachers will go beyond this; but we do not see how to specify such activities as part of a common *mathematics* curriculum.) One of the reasons for strengthening the core expectations in elementary mathematics is the hope that *other subjects* may then have the confidence to build on the resulting techniques and inject a more mathematical approach into some of their own basic material.

9. Ratio, proportion, the unitary method

Elementary mathematics comes into its own (and needs to be seriously *taught*!) as soon as we take the step from addition to *multiplication.* The simplest, and in many ways the most valuable, applications of school mathematics—to life, to science, and to mathematics itself—occur when two quantities are related in such a way that if one quantity doubles, or triples, so does the other: that is, where the numerical values of the two related quantities have a **constant ratio**. Two quantities that vary in such a way as to preserve a constant ratio between their values are said to be "in **proportion**". If £100 is worth the same as \$150, then we can be sure that £200 is worth \$...; and if 1 kg is the same as 2.205lbs, then 2kg is the same as ... lbs.

A *proportion* problem typically involves *four* quantities—three of which are known: one knows one pair of corresponding values (say £100 is worth \$150), and one wants to know "How many \$ will one get for £768 (say)?"

if $\quad \pounds 100 \longrightarrow \$150$

then $\quad \pounds 768 \longrightarrow \$??$

Or inversely, if \$150 is worth £100, how many £ should one expect for \$1052 (say)?

This standard way of representing the four pieces of information in a *proportion*—three known values and one generally unknown—is referred to here as the *rectangular template* for displaying *proportion* problems.

Proportion questions may eventually be answered directly—by extracting the ratio, or fractional multiplier $\frac{768}{100}$ from two of the quantities, and applying it to the third known quantity to find the required value

$$150 \times \frac{768}{100} = \ldots.$$

But for most students, the **unitary method** provides an essential stepping stone *en route* to this general method—a stepping stone to which one can appeal in any setting to re-explain, or to reinforce the general method: given three of the four relevant values, instead of using the 'magic multiplier' immediately, we use two of the values—here £100 and \$ 150—to calculate

- first how many dollars corresponds to £1 (the *unit*)
- then to multiply the answer (\$1.50) by 768 to get the number of \$ that correspond to 768 × **£1**:

$$768 \times \$1.50 = \$\ldots$$

Thus

$$\begin{array}{lll} \text{if} & \pounds 100 & \longrightarrow \$150 \\ \text{then} & \mathbf{£1} & \longrightarrow \$1.50 \\ \text{so} & \pounds 768 & \longrightarrow 768 \times \$1.50 = \$\ldots \end{array}$$

10. Inverse problems, simplification

We have also made occasional reference to **inverse** problems. These arise on several different levels.

The simplest examples of *inverse* problems arise from the fact that mathematical operations often come in what we call 'direct-inverse' pairs: for example,

- the act of adding A to B is reversed when we subtract A from the answer;
- that act of multiplying A by B is reversed when we divide the answer by B;
- the act of calculating powers is reversed when we 'extract roots';
- the act of multiplying out brackets is reversed when we try to factorise.

The first operation in each pair tends to be relatively easy to implement (e.g. addition, multiplication, taking powers, or multiplying out brackets). In contrast, the 'inverse' operation (subtraction, division, extracting roots, factorising) is more challenging. The underlying idea of the 'inverse' operation can be grasped fairly easily. But *systematic* success with the *inverse* operation requires one to 'juggle possibilities' using the direct operation; so pupils need to be completely fluent in implementing the direct operation before they stand a chance of being *systematically* successful with the corresponding *inverse* operation (e.g. addition must be fluent before one tackles *systematic* subtraction; multiplication must be fluent before one tackles *systematic* division; etc.). The process of reversing the direction of calculation, and working backwards to identify the required output for the inverse operation, presumes confidence and accuracy in handling the direct operation—

which is one reason why the *inverse* operation often serves as an excellent diagnostic as to whether the direct operation has been mastered sufficiently fluently. Moreover, whereas the direct operation is often mechanical and deterministic (i.e. one learns to turn a handle to produce the answer—as with addition, or multiplying out brackets), the *inverse* operation (as with subtraction, or factorising) is far more challenging.

Because it is so much harder to master *inverse* procedures, examiners who are concerned to achieve improved 'success rates' tend to limit the number of questions that test *inverse* processes; and where such questions are set, the need to 'juggle possibilities' is often removed by providing the necessary intermediate steps. Teachers then draw the obvious conclusion, and spend relatively little time exercising the more demanding inverse technique. Unfortunately, *it is often the inverse procedure that is the most important for subsequent mathematical progress.*

The characteristic phenomenon of *inverse* processes—namely "juggling possibilities", and "working backwards"—gives rise to a whole universe of *inverse problems*, which are invaluable as problems in their own right, and are useful in testing whether basic techniques have been mastered in a sufficiently robust way. Two typical classes of examples are:

1. "I'm thinking of a number" problems, which require pupils (in Key Stage 2) to *reverse* the processes of mental arithmetic.
   "I'm thinking of a number.
   If I double my number and add 7*, I get* 25*.*
   What was my number?"
   Later such problems will be solved mechanically by setting up an equation. But in Years 5–7 the more primitive approach is an excellent way of developing mental arithmetic in the form of *inverse* problems, and of training pupils to store, and to juggle possibilities in their heads.

2. In "missing digit" problems a completed calculation is given in standard column format—but with some of the digits missing. Pupils then have to use what they know about arithmetic, or about the relevant algorithm, to reconstruct the missing digits.

Inverse problems have another valuable feature. Whereas much school mathematics is relatively routine, *inverse* problems are more interesting—requiring a kind of 'detective' work, where pupils need to identify and use what they know in slightly non-standard ways.

One general instance of *inverse* techniques pervades the whole of mathematics—namely the art (and habit) of **simplification.** A given fraction, such as $\frac{1}{2}$, can be routinely (i.e. 'directly') transformed into equivalent fractions

$$\frac{1}{2}, \frac{2}{4}, \frac{3}{6}, \frac{4}{8}, \frac{5}{10}, \ldots$$

But what is ultimately needed is for pupils to recognise instantly that $\frac{18}{36}$ belongs to the same family—not because they have learned it by heart, but because they have mastered the *inverse* art of *cancellation.*

The art of *simplification* lies at the heart of elementary mathematics. The process whereby

- we make friends with certain 'standard forms' (such as $\frac{1}{2}$, or $(n+1)^2$), and
- then habitually and routinely rewrite wilder-looking expressions in terms of these standard forms

is what makes it possible to carry out calculations, to recognise when two different-looking algebraic expressions are in fact *identical* (i.e. when one can be transformed into the other using the rules of algebra), and in general, to analyse simple questions mathematically.

11. Structural arithmetic

One feature of mathematics teaching at all levels is the need to re-visit topics and methods which have been previously covered. Even where the material has previously been well taught, much may be gained by spending time revising and strengthening vaguely familiar language, ideas and methods—provided it is done in a *forward-looking way.* Indeed, for those who failed to grasp a topic at the first encounter, subsequent re-visiting and revision is essential if they are to progress. But those who appeared to understand things the first time round also have much to gain from re-visiting basic material *in the right spirit.*

The 2003 and 2007 results of TIMSS (a 4-yearly study of school mathematics in different countries) revealed a marked improvement in average scores among Year 5 pupils in England. The natural first response was to see this as constituting resounding support for the impressive efforts that had gone into the early *Numeracy Strategy.* But closer inspection (for example, of those problems where English pupils performed less well) suggests that we need to change our focus—especially later in Key Stage 2 and early Key Stage 3. This impression is reinforced by the fact that the apparent improvement in average Year 5 performance was not reflected in a corresponding improvement at Year 9 (even though the 2007 Year 9 sample was from exactly the same cohort as the 2003 Year 5 sample).

Our conclusion is that at Key Stage 2 pupils need something more than indiscriminate practice in mental calculation. When developing children's confidence with mental calculation, it may seem sensible to encourage them to use and to talk about *their own* methods—as long as they are effective. But as the numbers being handled get larger (or when, during Key Stage 3, the algebraic structure of the expressions to be manipulated gets more complicated), most of the expressions one meets cannot be so easily evaluated or simplified. Progress in mathematics then depends on mastering—and learning to exploit—the *algebraic rules* which sometimes allow one to *simplify* unexpectedly. So from Key Stage 2, calculation has to begin to move beyond bare hands evaluation, and to concentrate on developing flexibility in looking for ways to exploit *place value* (as in $73+48+27 = \ldots$), and an awareness of the algebraic *structure* lurking just beneath the surface of so many numerical or symbolical expressions (as in $3 \times 17 + 7 \times 17 = \ldots$, or $\frac{6\times15}{10} = \ldots$, or $16 \times 17 - 3 \times 34 = \ldots$, or $6(a-b)+3(2b-a) = \ldots$)—a theme which we refer to here as **structural arithmetic**.

There is nothing wrong with encouraging pupils to develop and use their own methods provided teachers understand when these methods should be left behind! Re-visiting and reviewing earlier material is essential

– to help pupils understand the need to loosen their grip on 'backward-looking' methods (which may have worked up to that point, but which often have limitations),

and

– to move them on to *'forward-looking' methods*, which may at first feel unfamiliar (and hence less congenial), but which are essential if pupils are to progress to more demanding material.

When faced with the need to solve a problem or to perform a calculation we all tend to use a method that we are comfortable with. But such methods are likely to be 'backward-looking', in that they come from the world that we already know, and so are limited by the mathematical universe we may have inhabited up to that point. By allowing pupils to continue using methods they are comfortable with, we often stifle their subsequent mathematical growth. So pupils need to be pushed to learn new *forward-looking* methods so that they can make subsequent progress. For example, many young children solve problems involving small numbers by counting on their fingers; but if they cling to this approach, they will come unstuck when faced with more demanding calculations.

From Key Stage 3 onwards these same observations lead one to focus

– on the general art of simplification (especially in working with fractions and fractional expressions, but later with surds and algebra),

and

– on the various settings for calculating-with-letters (using formulae, solving equations, working with algebraic expressions, and eventually with functions).

By the end of Key Stage 4 pupils need to be as fluent in handling 'algebraic expressions' (a term which includes unevaluated *numerical* expressions written *in algebraic form*—using powers, roots, and quotients) as they are at the end of Key Stage 2 in evaluating mental and written calculations. And as pupils move through these phases, they need to master the underlying rules which often allow them to *simplify*, and so to recognise that what may at first look unfamiliar is often something much more friendly in disguise.

12. Divisibility tests

Divisibility tests have long been the poorly understood *Cinderellas* of school mathematics. They are in fact a simple, satisfying, and enlightening instance of *structural arithmetic*.

The fact that *multiples of* 10 are precisely the integers having "units digit $= 0$" is an evident consequence of *place value*.

Any integer is equal to 'a multiple of 10' plus its 'units digit'. Since the 'multiple of 10' is also a 'multiple of 2' ($10k = (2 \times 5)k = 2 \times (5k)$), the original integer is a 'multiple of 2' precisely when its units digit is a multiple of 2—that is, when the integer ends in 0, 2, 4, 6, or 8.

Similarly, any 'multiple of 10' is also a 'multiple of 5' ($10k = (5 \times 2)k = 5 \times (2k)$). So an integer is a 'multiple of 5' precisely when its units digit is a multiple of 5—that is, when it ends in 0, or 5.

Any integer is equal to a multiple of 100 plus the two-digit number formed by its tens and units digits; so multiples of 100 are precisely the integers having both tens and units digits $= 0$. A multiple of 100 is also a 'multiple of 4' ($100k = (4\times25)k = 4\times(25k)$); so an integer is a 'multiple of 4' precisely when 'the number formed by its *last* ***two*** *digits*' is a multiple of 4.

Similarly multiples of 1000 are precisely the integers having hundreds, tens and units digits = 0. Since any multiple of 1000 is also a 'multiple of 8' ($1000k = (8 \times 125)k = 8 \times (125k)$), an integer is a 'multiple of 8' precisely when 'the number formed by its *last **three** digits*' is a multiple of 8.

This shows how the rules for spotting multiples of 2, or 4, or 5, or 8, or 10 derive from the *place value* system for writing numbers.

The divisibility tests for multiples of 3, or of 9, depend on the *place value* system in a more interesting way, which obliges us to think about the *algebraic* structure of the place value system. The key here lies in the fact that

$$10 - 1 = \dots, \quad 100 - 1 = \dots, \quad 1000 - 1 = \dots$$

etc. are all multiples of 9. This is a special case of the beautiful factorisation

$$x^n - 1 = (x-1)(x^{n-1} + x^{n-2} + x^{n-3} + \cdots + x + 1).$$

So any integer such as 12345, can be deconstructed into

$$\begin{aligned}\mathbf{12345} &= 1 \times 10000 + 2 \times 1000 + 3 \times 100 + 4 \times 10 + 5\\ &= 1 \times (9999 + 1) + 2 \times (999 + 1) + 3 \times (99 + 1) + 4 \times (9 + 1) + 5\\ &= (1 \times 9999 + 2 \times 999 + 3 \times 99 + 4 \times 9) + (\mathbf{1} + \mathbf{2} \ + \mathbf{3} + \mathbf{4} \ + \mathbf{5})\end{aligned}$$

The first bracket is always a multiple of 9—and hence also a multiple of 3. So for 12345 to be a multiple of 3, the second bracket—that is, its *digit-sum* '1+2+3+4+5'—must be a multiple of 3 (which it is!). And for 12345 to be a multiple of 9, the second bracket—that is, its *digit-sum* '1+2+3+4+5'—must be a multiple of 9 (which it is not). This yields a simple (and intriguing) test for divisibility by 3 and by 9.

The test for divisibility by 6 is slightly different: an integer is divisible by 6 precisely when it is divisible by 2 **and** by 3. Similarly, an integer is divisible by 12 precisely when it is divisible by 4 **and** by 3 (where it is important that $hcf(3, 4) = 1$).

Divisibility by $11 = 10 + 1$ depends on a simple variation of the reasoning for divisibility by $9 = 10 - 1$. The key here lies in the fact that

$$10 + 1 = 11, \quad 100 - 1 = 99, \quad 1000 + 1 = 1001,$$

etc. are all multiples of 11.

There is no difficulty in checking that the even terms in this sequence

$$99 = 11 \times 9, \quad 9999 = 11 \times 101, \quad 999999 = 11 \times 10101,$$

etc. are multiples of 11.

To show that the odd terms 11, 1001, 100001, etc. are also multiples of 11 we use the beautiful factorisation for odd powers n

$$x^n + 1 = (x + 1)(x^{n-1} - x^{n-2} + x^{n-3} - \cdots - x + 1).$$

Any integer such as 71819 can then be deconstructed into

$$\begin{aligned}\mathbf{71819} &= 7 \times 10000 + 1 \times 1000 + 8 \times 100 + 1 \times 10 + 9\\ &= 7 \times (9999 + 1) + 1 \times (1001 - 1) + 8 \times (99 + 1) + 1 \times (11 - 1) + 9\\ &= (7 \times 9999 + 1 \times 1001 + 8 \times 99 + 1 \times 11) + (\mathbf{7} - \mathbf{1} + \mathbf{8} - \mathbf{1} + \mathbf{9})\end{aligned}$$

The first bracket is always a multiple of 11. Hence, for 71819 to be a multiple of 11 the second bracket—that is, its *alternating* digit-sum '$7 - 1 + 8 - 1 + 9$'—must be a multiple of 11 (which it is!).

13. Symbols: expressions, formulas, equations, identities

It is hard to be succinct about what needs attention in the teaching of elementary *algebra*, since much of what has been written on the subject in the last 40 years has diverted attention from the central issues—with evident consequences.

Elementary algebra is about translating pupils' experience of *structural arithmetic* into a mastery of working with letters—in expressions, formulas, and equations, and in working and calculating in the coordinate plane.

Just as primary arithmetic later evolves into *structural arithmetic*, so once elementary algebra has been mastered, some of its ideas will later be reformulated in terms of **functions**. But nothing is gained and *much is lost if this shift of focus is made too soon.*

Algebra as we know it emerged after 1630 AD. It revolutionised the ease with which mathematical calculations could be carried out and communicated. Within 40 years this revolution gave birth to analytic (coordinate) geometry, and to calculus! It allowed material which had previously been poorly understood even by specialists to become the birthright of Everyman. In short, just as place value brought number and calculation within reach of the masses, so elementary algebra gave us a language in which the world of mathematical ideas became accessible to ordinary people.

Elementary algebra emerges from a shift in the approach to calculation: in place of blind, or idiosyncratic, computation, one emphasises *structural arithmetic*; instead of grinding out answers, one looks for 'inner structure' (as in $148 + 73 - 48 = \ldots$, or $17 \times 3 + 17 \times 7 = \ldots$, or $99 \times 6 \div 3 = \ldots$). One gradually realises that such *structural arithmetic* encapsulates rules that transcend the particular calculations, and that can be expressed in symbolic form ($(a + x) + b - x = \dots$, $17a+17b = 17(a+b)$, $a \times 2c \div c = 2a$): that is, one can write, and *simplify*, **expressions** by treating symbols as though they were numbers. One can then translate the relevant parts of one's calculational experience with numbers into a universe where one does arithmetic with *symbolic expressions*—routinely simplifying as one proceeds.

The letters in this new 'calculus of expressions' have no particular meaning. But a parallel application of the same arithmetico-algebraic principles arises when one learns to handle **formulas**, which express the relationship between variables which *do* have meanings (such as: $A = l \times b$; $C = 2\pi r$; $A = \pi r^2$). Again the symbols can be moved around 'as if they were numbers' to deduce other useful formulas (such as $r = \frac{C}{2\pi}$).

Later one discovers how *inverse* arithmetic problems from Key Stage 1 and 2, such as those based on "I'm thinking of a number ...", can be routinely solved by setting up and solving simple **equations**—where the solving process depends on treating numbers and symbols (i.e. coefficients and unknowns) as 'calculational objects of the same kind'.

And at some stage, the growing fluency in manipulating and simplifying expressions gives rise to standard **identities**, such as

$$(a+b)^2 = a^2 + 2ab + b^2,$$

and

$$a^2 - b^2 = (a-b)(a+b).$$

These standard *identities* then make it possible to recognise an important general phenomenon: unfamiliar expressions are *identical* whenever one expression can be transformed into the other using the rules of algebra; that is, they are 'identically equal', where—unlike *equations*—the equals sign holds true for *all* values of the variables. (Failure to transform one expression into another does not prove that they are *not* identical. The easiest way to prove that two expressions are non-identical is to find particular values for the variables so that, when one substitutes these values into the two expressions, one gets different outputs.)

14. Sequences: intrinsic definitions, closed formulas

A **sequence**

$$x_1, x_2, x_3, x_4, x_5, \dots$$

is a way of presenting an endless amount of information in a single list. There are two quite different ways of specifying the terms of such a *sequence.*

The first, and most primitive, way is by giving the first few terms and then specifying a *term-to-term* rule (or a 'recurrence relation') that tells you how to work out the next term from the ones you already know. For example,

- $x_1 = 3$, $x_{n+1} = 2x_n$ defines the sequence 3, 6, 12, 24, 48, ... ;
- $x_1 = x_2 = 1$, $x_{n+1} = x_n + x_{n-1}$ defines the sequence 1, 1, 2, 3, 5, 8, 13, ...
- $x_1 = 2$, $x_{n+1} = 3x_n - 2^n$ defines the sequence 2, 4, 8, 16, 32, ...

Although this first approach allows you to continue the sequence a far as you like, and determines the 10^{th}, the 100^{th}, and the 1000^{th} terms uniquely, its weakness lies in the fact that it may not be easy to obtain a proven **closed formula** which gives the n^{th} term of the sequence as a formula in terms of n.

- In the first example, the first term $x_1 = 3$ is doubled $n - 1$ times to get the n^{th} term, so $x_n = 3 \times 2^{n-1}$
- In the second example, it is easy to generate more and more terms, but it is quite unclear how to write the n^{th} term as a *closed formula* in terms of n.
- In the third example, it is easy to guess that the *closed formula* for the n^{th} term must be $x_n = 2^n$, but it is not at all clear how to prove that this is correct.

In short, a term-to-term rule is easy to use, but it is inefficient: the only way to work out the 50^{th} term is to first work out the previous 49 terms; and there is no way to reason in general about the *n^{th} term.*

The second (and much more powerful) way to specify a sequence is by a *position-to-term* rule, which tells you how the n^{th} term can be calculated directly in terms of n.

A position-to-term rule may also define a sequence **intrinsically**, with the n^{th} term being defined to be a number which can be calculated from some figure or geometrical configuration. For example,

1 point on a circle gives rise to 0 chords
2 points on a circle give rise to exactly 1 chord
3 points on a circle give rise to 3 chords
etc..

The position-to-term rule tells us how to find each term; we can therefore generate the sequence 0, 1, 3, 6, 10, 15, ... whose n^{th} term is equal to the number of chords created by n points on a circle. The sequence is well-defined, but there is (at this stage) neither a term-to-term rule nor a *closed formula*, and any claims we might make about how the sequence behaves must be deduced from the given geometrical definition of the terms.

15. Exactness

In this curriculum, references to calculation, to reasoning, and to proof are occasionally prefixed by the word **exact**. Wherever this word appears the attentive reader is encouraged to pause, and to consider why it has been inserted.

Mathematics used to be known as "the *exact* science". Unlike disciplines that work with real data or objects, mathematics studies a world of idealised, *mental* objects. The objects studied in elementary mathematics often have their roots in the world of human experience; but they become mathematical only when the underlying ideas are abstracted from these roots. For example,

– *numbers* have their roots in experience; but
– they soon become "mental objects" with exact properties, and can be manipulated in the mind.

In much the same way, a sheet of A4 paper, or a door, may serve as a suggestive model for a rectangle; but

– a mathematical rectangle is a perfect mental object, whose diagonals are *exactly* equal—their length being given *exactly* (in terms of the two sides) by Pythagoras' theorem.

The mathematical universe consists of *imagined* objects, which are precisely defined, and hence uniquely knowable. In particular, mathematics, or "the art of *exact* calculation", belongs to a completely different conceptual universe from the practical world in which one might "draw a scale diagram of a rectangle and *measure* the approximate length of a diagonal". It should be one of the main goals of any curriculum to convey the distinction between the *exact* world of mathematics, and the approximate world where mathematics is used and applied, and to show pupils why we have no choice but to engage with the *exact* world of mathematics—even to solve the most practical of problems.

In the everyday *real* world of experience, measurements and ideas are inescapably "fuzzy". The fact that the world of mathematics operates on *ideal* objects allows its ideas, its notation, its methods of calculation, and its processes of logical deduction to be *exact*: this guarantees that the answers and conclusions produced in mathematics are as reliable as the information that was fed into the relevant calculation or deduction. The importance of this aspect of elementary mathematics has become blurred in recent years.

The process of developing internal methods for calculating with *exact* mental objects (whether numbers, symbols, shapes, or functions) is much the same today as it ever was, and is rooted in mental work and written hand calculation. Once these ideas are suitably embedded in the mind, calculators and other tools have much to offer: but initially, much of the learning process proceeds more naturally without such distractions.

16. Estimates, approximation

We have seen how mathematical *exactness* is quite different from real-world *precision*. *Exactness* in mathematics allows no scope whatsoever for error; indeed, in an *exact* calculation any error of any kind undermines the validity of the whole process. In contrast, the idea of "precision" recognises that, outside mathematics, all measurements incorporate a degree of error, and so are **approximate**. When applying mathematical methods to values which are only known *approximately*, we need to know

(a) the maximum extent of potential error in the given data, and
(b) how these potential errors accumulate when one carries out *exact* calculations with numbers that are only known *up to this level of accuracy*.

When adding or subtracting approximate numbers, the errors in the data *add up*. Given two lengths of 2.15cm and 1.75cm—each correct to within 0.05cm—their calculated *difference* of 0.40cm is only correct to within 0.1cm, so could actually be as low as 0.30cm or as high as 0.50cm. That is, an error in the inputs of 1 part in 40 gives rise to a possible error in the output of 1 part in 4!

When multiplying or dividing the story is more complicated. The area of a rectangle measuring 150cm by 120cm, where each measurement is accurate to within an error of 0.1cm, is equal to $150 \times 120\text{cm}^2$, or 18000cm^2—but only to within 27.01cm^2).

The art of making **estimates**, or *approximate* calculations, is more subtle than is often thought. It requires

(a) a robust fluency in *exact* calculation,
(b) a willingness to change *global* units intelligently (replacing the

given units by larger or smaller 'blocks' so as to make the eventual calculation more manageable), and

(c) an ability to make sensible *local* approximations (to find the approximate size of one of these new 'blocks' and to estimate the number of 'blocks').

For example, faced with a multiplication such as 35941×273:

1. we need to revert to the meaning of multiplication in order to think of this as 35941 blocks of 273;
2. then appeal to our understanding of how the exact calculation operates on the given (exact) inputs to see that 35941 blocks (each of size 273) could be advantageously interpreted as "slightly more than $\mathbf{33\frac{1}{3}}$ *thousands*", and
3. to compensate this underestimate of the number of blocks by interpreting the block size of 273 as "slightly less than **3** *hundreds*".

Hence the required answer is "approximately **100** hundred thousands", or 10 million.

Similarly, in seeking to estimate the size of a large crowd, one may divide the whole into a number of blocks of more-or-less the same size, count the number in a given section of the crowd relatively accurately (for example, by counting the number of rows and the number in each row), and then multiply the answer by the number of blocks. A striking example occurs in Herodotus, *The Histories*, Book 7:

> 'As nobody has left a record, I cannot state the precise numbers provided by each separate nation [towards the army that Xerxes was leading against the Persians], but the grand total, excluding the naval contingent, turned out to be 1 700 000. The counting was done by first packing ten thousand men as close together as they could stand and then drawing a circle around them on the ground; they were then dismissed and a fence, about navel-high, was constructed round the circle; finally the other troops were marched into the area thus enclosed and dismissed in their turn, until the whole army had been counted.'

For pupils to master the art of *approximating* arithmetical calculations in integers, they first need to master the art of *exact* calculation. Only then can they use their knowledge of exactness as a fulcrum for thinking precisely about more elusive *approximation*, or *estimation*. And when they come to analyse the errors introduced by such approximations, they will find that this is done via the *exact* calculations of elementary algebra. Thus, even when seeking to transcend the inherent exactness of arithmetic by developing the art of making *estimates*, there is no escape from the maxim:

Mathematics is the science of ***exact*** *calculation.*

17. Euclidean geometry: a 'formal' approach

As with all aspects of elementary mathematics, there is no "royal road" to success in geometry. The approaches adopted since the 1970s have introduced all manner of delights, which we should hesitate to discard. But they have singularly failed to produce school leavers able to analyse configurations in two- and three-dimensions.

During this period certain teachers and authors have continued to insist, and to demonstrate, that the most effective framework within which ordinary students can apprehend and learn to 'calculate exactly' with geometrical information is one that analyses more complicated figures in terms of triangles and circles. This is the thrust of the **Euclidean** framework adopted here. *Informal* work at Key Stage 1 to apprehend shapes and patterns in 2D and 3D moves on at Key Stage 2 to prepare the ground for a 'more formal' treatment at Key Stages 3 and 4. The *informal* work at Key Stages 1 and 2 starts by drawing, measuring, and calculating angles, lengths, areas, and volumes, in order to develop the ideas and language that will be needed for the later **formal** version of **Euclidean geometry**.

In particular, Key Stages 2 and 3 below make cryptic reference to *angle-chasing*. This is a shorthand for any activity in which a 2D configuration of lines and line segments is specified, with the sizes of certain angles given, from which the sizes of other angles are to be logically determined (by reasoning and calculation, not by measuring). For example, if the required angle were the third angle in a triangle whose other two angles were given, then the required angle could be immediately deduced. In general the size of the required angles may not be immediately deducible, but may require one to first calculate certain intermediate results. That is, *angle-chasing* is a restricted (geometrical) framework for a whole class of problems that are *multi-step*, and that are also deductive exercises in the basic angle properties (angles at a point, angles on a straight line, vertically opposite angles, angles in a triangle—and later alternate angles).

Key Stage 3 then works to establish the beginnings of a *formal* framework rooted in

– points and lines (which stretch endlessly in each direction),
– line *segments* (that part $\underline{AB}$ of the line AB which lies *between* two points A, B on a given line),
– angles ($\angle ABC$),
– triangles (where $\triangle ABC$ is an *ordered* triple—not just as a perceived "shape"),
– circles, and
– *congruence* of triangles (SAS, SSS, ASA, RHS).

One then learns to 'calculate' and to reason exactly and reliably within this *formal* framework.

Other approaches to elementary *Geometry* have been tried; but they all have significant drawbacks. On reflection, we have concluded that the most effective beginners' introduction to the analysis of configurations in 2D and 3D is via a simplified version of the classical Euclidean approach—based on triangles as fundamental building blocks. An important bonus in this approach is that it provides access for pupils of all ages to the delights of 'exact logical calculation'—as in *angle-chasing*, and *proof*. The approach has a proven pedigree, and combines simplicity, clarity of language, opportunity for hands-on construction, and a natural progression to a *formal* framework, within which results can be proved at Key Stage 4. But the key point is that, once one accepts that non-obvious results should be *proved*, the logical structure—and hence the details of the approach—become part of the *What?* rather than the *How?*

18. Ruler and compass constructions

Part of our development of *Geometry* in Key Stage 3 moves

– from preliminary measuring work with rulers and protractors at Key Stage 2
– to a simpler hands-on geometrical framework which avoids measuring altogether, in which the familiar 'measuring ruler' becomes a *straightedge* (that is, a mere straight-line-drawer), and the focus switches from measuring to 'equality', or *congruence*, of line segments.

We stick to the tradition of referring to these latter constructions as **ruler and compass constructions**—even though the ruler is being used as an **ideal** *mental straightedge* (and its crude, approximate markings play no role). Given two points A and B, the **ruler** is simply a way of physically capturing the idea of the *line segment* $\underline{AB}$ and the *line*

AB determined by these two points. And given a point O (as centre) and another point P, the **compasses** are a way of physically capturing the **ideal** construction of the *circle with centre O and passing through P*. That is, though the two instruments are being used to perform hands-on approximate constructions, their true function is to combine hand, eye, and brain in a way that indicates *imagined ideal constructions*, that would be *exact* if performed with 'heavenly' straightedge and compasses.

Ruler and compass constructions offer a natural psycho-kinetic embodiment of the simplest parts of *formal* geometry (for example: bisecting a given line segment $\underline{AB}$; or the SSS-compass-construction of a triangle given the three side lengths, which motivates the SSS-congruence criterion). The *proofs* that these basic constructions do what they claim use only the basic (congruence) principle, and so provide a gentle, but striking introduction to the transition from practical hands-on experience to *formal* geometry.

19. Proof; line-by-line format

The essence of elementary mathematics lies in the twin facts
- that its domain is restricted and abstract,
- but within that limited domain, the knowledge that it delivers is ***certain***—*objective*, not subjective.

Thus all pupils should expect the methods of elementary mathematics to make logical sense, and should come to understand that its procedures are exact and deductive (rather than approximate and inferential).

It follows that even the simplest calculation should be presented in a way that constitutes a **proof** that the answer to the original problem is undeniably what emerges at the end of the calculation. This is best conveyed by laying out calculations and deductions **line-by-line**,
- with the given information, and any symbols representing 'unknowns' declared at the outset,
- with each fresh step on a new line (and any explanation given alongside)

and
- with the final answer clearly displayed at the bottom.

The sequence of successive steps can then be grasped as a single chain of reasoning, in which each step follows clearly from those which went before.

This format can be used for simple calculations, for presenting the solution to an angle-chasing problem, for setting up and solving an equation, for proving that two algebraic (or trigonometric) expressions are identical, for justfiying a ruler and compass construction, and for laying out a formal **proof** (such as, that the angle in a semicircle is a right angle).

20. Probability, populations, samples

As we have repeatedly said, the 'elementary mathematics' which is covered by this draft curriculum may be largely summarised as *the art of exact calculation* with numbers, symbols, geometrical entities, etc. To find "the height of Nelson's column", it is natural to assume it has a value, and then use the methods of elementary mathematics to calculate it using other known facts (e.g. the properties of similar triangles). That is, an object to which this 'art of exact calculation' applies may be represented as a mathematical entity—in this case, by a line segment, whose length has a fixed (if as yet unknown) value, which remains constant throughout any subsequent calculation. Such entities are relatively tame, and static, and can be imagined relatively easily.

However, some numerical data is more elusive than this—though the fact that it is clearly still numerical (in some sense) may tempt us to overlook its more elusive character. Consider, for example, "the height of a UK adult male in 2010". Each concrete instance of such data (choose one adult UK male, and measure his height) gives rise to a *single value*—the height of that particular individual. So it is superficially *like* "the height of Nelson's column". However, the object of thought is *not* this individual value, but the *totality* of individual heights, and the way these individual heights are *distributed* throughout the whole population of "UK adult males in 2010". This object of thought is multi-layered: there is a **population** S (the set of UK adult males in 2010), with each member having an attached *number* (his height); this attached number varies as one varies the choice of individual, and does so in such a way as to give rise to a *distribution* of possible values, where each "height" occurs with its own *frequency*, or **probability**. Later, these multi-layered objects will be formalised as *random variables*, and captured via *distributions*: all we need to note here is that they are clearly harder to pin down than the numbers studied in the rest of elementary mathematics!

This is just the easy part of the story. The hard part is that we rarely know this background *distribution* precisely, and the goal is to draw inferences about it on the basis of some more-or-less representative *random* **sample**! (The word 'random' deserves a whole mini-essay of its own; but it indicates that the sampling is done in a way that avoids giving a systematically false impression of the population being sampled.) That is,

– the specific entities of elementary mathematics are here replaced by *distributions*
– which involve a range of possible values—each with its own frequency
– the background *distribution* being generally unknown—and all we know is information about some *sample*.

The goal is then to decide what we can infer about the (unknown) background *distribution*. This is an important art. But it is very different from—and conceptually much more demanding than—the mathematics of 'single-valued measures', or functions, studied in the rest of elementary mathematics.

In the relatively tame world of elementary mathematics we have already highlighted the difference between *direct* calculation, where the answer can be ground out deterministically, and *inverse* problems (e.g. "Can 1729 be written as a sum of two cubes?") whose solution forces us to 'work backwards' from some 'output' in search of some direct calculation that might give rise to the given data. The art of analysing statistical data mathematically is an important instance—and a much more subtle instance—of an *inverse problem*: not only are the objects of the relevant *direct* statistical calculations more subtle than those we meet in the rest of elementary mathematics, but *handling data* is useful precisely because statistical problems are *inverse* problems: we typically know only selected information (from some presumed *random sample*), and we need to assess what can be inferred from this *sampled data* about the unknown background *distribution* of the whole *population*—and what degree of confidence we may attach to such an inference.

This material plays such an important role in modern society that it is natural for educators to try to find ways of introducing pupils to the underlying ideas. Much excellent work in this direction has been done. And though it is not easy to summarise the experience of the last 25 years, it is fair to say that the rhetoric has been consistently ahead of the reality. So there are many outstanding issues which a draft curriculum has to weigh up and resolve as best it can. The two most important questions concern

- the age, or prerequisite maturity required, before worthwhile mathematical analysis of statistical material can be handled effectively;
- how much time is needed to prepare the way for such worthwhile analysis, what progress can be made at a given stage, and what core material should give way to make time available at that stage.

Roughly speaking, the difficulties implicit in the first question, and the determination of previous curricula to address *Handling data* very early, have forced the content of what is taught under this heading to consist mostly of low-level *descriptive statistics*. There is significant value in using common sense to extract simple information from statistical data, and to clarify common errors and abuses. But one has to assess how much time should be devoted to such material, and how much of this time should be taken out of the time allocated for *mathematics*—given that serious *mathematical* analysis of statistical problems remains well out of reach. Such concerns have proved especially challenging in formulating the current draft. Methods often cannot be understood at the level they are used; and are then used inappropriately (see, for example, *Mathematics: made to measure* p.21). The fact that statistical methods–even when used correctly—are often presented only in cookbook style, without clear justification, contradicts the intended character of the rest of this curriculum, in which every effort has been made to insist on *meaning* and *understanding*, and where terms and ideas have been included only if they make significant parts of elementary mathematics accessible to pupils' *own* analysis.

On reflection it seems clear that in recent years the position of *Handling data* at this level has been inflated, and that areas which are essential prerequisites for beginners have suffered a loss of time and emphasis as a result—as was clearly indicated in the analysis and recommendations of the Smith report *Making mathematics count* (2004).

Given the ubiquity of statistical data, some understanding of the associated problems deserves attention. But it remains unclear how this experience should be embedded within the wider curriculum, and how much of it, and which aspects, are best treated in the time allocated to *mathematics* (and at what stage). This draft curriculum struggles to take account of the subtlety of the material (which often seems to be ignored by enthusiasts), and to reflect the urgent need to improve the general level of attainment in much more basic material. When professional debate eventually engages with the underlying problems of priorities and timing, we would hope to improve on the current draft.

21. Calculators

The previous section explained why this draft interprets *Handling Data* at this level as an application of number, rather than as a strand in its own right. In the same spirit we should perhaps explain the position of this draft with regard to **calculators**. *Calculators* are essential participants in the mathematics *classroom*; but they are not part of the *mathematics* that is being taught.

When meeting a new idea, a beginner is usually better off working with numbers that have been chosen so that they can be worked with *by hand*, or juggled in the mind. Nevertheless, there are clearly cases where preliminary experimental work can be designed to make good use of a calculator. There are also contexts in which a calculator is indispensible—such as working with trig functions and inverse trig functions when 'solving triangles'. And there are plenty of settings in which, once the underlying principle is securely understood, the availability of a calculator allows pupils to experience the full power of this or that method (as when deciding whether or not $N = 10001$, or $N = 100001$ are prime, where the square root test can be used very effectively given a calculator and a list of primes up to $\sqrt{N}$). But in general, if ideas and methods are to be understood and to take root in the mind, pupils need more time and effort working *by hand* than has been admitted. So, though calculators have a significant role to play, and though the power of calculators and computers vastly increases the range of application and *ultimately* allows us to do far more than was the case 50 years ago, such black boxes have to be used with care if they are not to hinder, rather than to support, pupils' development.

Note: In the programme that follows, a small number of topics are included in curly brackets $\{\dots\}$. These are topics intended for $30-50\%$ of each the cohort, and are included in the hope that the assessment system will soon evolve to allow a "linked pair" of GCSEs *in series* rather than *in parallel* (that is, with a second paper which is more demanding).

Very brief versions

Key Stage 1: A very brief version

By the end of Year 2 pupils:

Counting, reading, and recording number

- use the language for numbers and quantities in everyday settings
- count accurately; read, write, and order numbers to at least 100; understand **place value**, know what each digit of any two-digit number represents, and know that the position of a digit determines its 'value'

Recalling facts

- have instant recall of addition and subtraction facts for numbers to 10; have instant recall of $\times 2$, $\times 5$, $\times 10$ multiplication tables, and derive corresponding division facts

Calculating

- use the language of simple calculations in everyday settings
- carry out mental and informal written calculations using the four operations of addition, subtraction, multiplication, and division
- recognise and use effectively the fact that subtraction is the inverse of addition, and in simple cases that division is the inverse of multiplication
- handle confidently two-digit addition and subtraction in standard column format

Describing shapes and measuring

- recognise, name, and describe properties of common 2D and 3D shapes
- measure and draw straight lines accurate to the nearest centimetre; estimate lengths and other quantities; tell the time to the nearest quarter of an hour, compare durations using standard units, and order events chronologically; use measuring instruments to measure length (cm, m), weight (kg), capacity (l); read and interpret scales to the nearest labelled division; use money

Solving problems, reasoning, and using language and symbols

- apply their understanding of number and arithmetic to work with measures and to solve **word problems**
- use mathematical language accurately; read and interpret text, diagrams, and symbols when solving problems; record their results clearly; explain their methods and reasoning

Key Stage 2: A very brief version

By the end of Year 6 pupils:

Place value

- handle place value to represent and order numbers to 10 000 and beyond; extend this to negative integers and decimals; work with decimals and measures involving tenths, hundredths, and thousandths; multiply and divide integers and decimals by 10, 100, 1000
- round integers, decimals, and measures to the nearest "ten", integer, or tenth

Recalling facts; correct use of language and symbols

- recall instantly addition and subtraction facts for numbers to 20; "know by heart" multiplication tables to 10×10 and corresponding division facts; factorise any two-digit integer; recognise primes and squares
- use mathematical language and notation correctly; understand that some statements are **exact** and can be clearly demonstrated

Structural arithmetic

- add and subtract positive and negative integers; multiply and divide positive integers; use place value and the structure of arithmetic to **simplify** calculations
- work flexibly with fractions and percentages; add and subtract simple fractions
- understand the order of operations and the use of brackets

Calculating

- add and subtract any two two-digit integers mentally, and three- and four-digit integers using **standard written column format**
- multiply and divide mentally a two-digit integer by any one-digit integer; complete written short multiplication and division of a three-digit or four-digit integer by numbers up to 12, and long multiplication of three-digit by two-digit integers

Geometry and measures

- copy simple figures; work with common 2D and 3D shapes; find unknown angles in simple figures; plot points with given coordinates
- measure and draw line segments accurate to the nearest millimetre, and angles to the nearest degree; calculate reliably with standard measures; find the areas of rectangles and shapes made from rectangles, and the volumes of cuboids and shapes made of cuboids

– use and calculate with money; tell the time to the nearest minute; read scales—interpolating between marks; convert between related units

Solving problems

– tackle and solve **word problems** and simple **multi-step** problems involving numbers, measures, and shapes; make sensible estimates; make **connections** between topics; explain their reasoning

Key Stage 3: A very brief version

Note: KS3 revisits important KS2 material—partly for revision, but mainly to see old material *from a new viewpoint* (e.g. extending the written algorithms to decimals; shifting the focus from bare hands mental methods to **structural arithmetic** and to simplify expressions; developing a deductive structure for known "facts"; etc.)

By the end of Year 9 pupils:

<u>Place value</u>

- handle **place value** to represent and order integers and decimals with up to six digits; multiply and divide by 10, 100, 1000; work with decimals and measures involving up to four decimal places
- round numbers and measures freely and flexibly

<u>Calculating</u>

- use multiplication tables freely to multiply and divide mentally in context
- compute with integers and decimals using standard column format
- compute with fractions; work flexibly with fractions, ratios, percentages

<u>Structural arithmetic</u>

- use **place value**, factorisation, and the algebraic structure of arithmetic to **simplify** and evaluate expressions including fractions and negatives
- use the algebraic equivalence of expressions to **simplify** calculations

<u>Simplification of algebraic expressions; solving linear equations</u>

- use unknowns and variables in context (formulae); use algebraic rules to **simplify** expressions, to collect like terms, to expand and to factorise simple expressions
- set up and solve a single linear equation in one unknown in complete generality; use the rules of algebra to "change the subject of", or to rearrange, equations and formulae; solve two simultaneous linear equations

<u>Geometry</u>

- measure and draw accurately; read scales; change units; understand and use basic formulae; find lengths, areas, and volumes for common 2D or 3D shapes; calculate reliably with standard measures
- plot points in all four quadrants; understand and work with linear

equations and straight line graphs; interpret gradient as a **ratio**, or ‘rate’
- use basic **ruler and compass constructions**, parallels, angles in a triangle, angle-chasing, congruence; establish a preliminary basis for 2D Euclidean geometry; prove and use *Pythagoras’ Theorem*

<u>Solving problems</u>
- tackle and solve **word problems**, and simple **multi-step** and **inverse** problems involving numbers, measures, symbols, and shapes
- use the **unitary method** to solve problems involving rates and **ratios**
- make sensible **estimates**; make **connections** between topics; explain their reasoning

Key Stage 4: A very brief version

Note: KS4 revisits important KS3 material—partly as revision, but also to interpret it afresh. For some pupils, this re-working and strengthening of KS3 material will be their main goal throughout KS4; others may supplement revision of KS3 material with a programme that covers selected parts of what is summarised here. The complete summary here is mainly relevant to those who expect to continue further academic studies beyond KS4.

By the end of Year 11 those who aim to complete the whole KS1-4 programme:

Number and measures

- handle (positive and negative) large numbers and decimals, with and without units, possibly expressed using powers or standard form
- move freely between fractions and decimals
- use rounding and **exact** arithmetic to work with **approximations**
- calculate **probabilities** in standard models; analyse sampled data

Calculating and simplifying

- compute with fractions; work flexibly with fractions, **ratios**, percentages
- solve problems involving **proportion** (including **the unitary method**)
- use algebraic structure and multiplication facts to simplify numerical expressions—including those involving fractions and powers
- calculate with surds and mixed surd expressions (without evaluating)

Algebra (expressions, formulae, equations, identities) and graphs

- use algebraic equivalence to **simplify** expressions
- know, use, and rearrange standard formulae
- work in all four quadrants; work with equations of straight lines in 2D
- solve any linear equation in one unknown; solve any pair of simultaneous linear equations in two unknowns; interpret the solutions graphically
- know and use standard quadratic identities
- sketch and analyse the graphs of quadratic functions
- solve any quadratic equation; solve easy simultaneous equations—one linear and one quadratic; interpret the solutions graphically

Geometry

- find lengths, surface areas, and volumes for common 2D and 3D shapes
- use basic trigonometry and the Sine and Cosine rules to 'solve triangles'
- understand and use basic **ruler and compass constructions**
- understand how congruence, parallels, and similarity provide a basis for **Euclidean geometry**; use these to derive results and to solve problems
- understand, prove, and use the basic properties of circles
- understand how scaling affects angles, lengths, areas, and volumes
- analyse standard 2D and 3D figures

Solving problems

- solve **word problems**; solve simple **multi-step** and **inverse** problems
- make **connections** between topics; write **proofs**; explain their reasoning
- calculate with standard and compound measures; work with 'rates'

Brief versions

Key Stage 1: Brief version

Key Stage 1: Breadth of study

During the key stage pupils should be taught the required *Knowledge, Skills, and Understanding* through:

a practical activity, exploration and discussion
b linking the language of mathematics with spoken and written English
c **learning key facts by heart**; learning to store tens and units temporarily in the mind (including as intermediate outputs in a longer calculation) to support the development of mental calculation strategies
d using mathematical ideas in practical activities, then recording these using objects, pictures, diagrams, tables, words, and numbers
e developing rich mental calculation strategies, and **standard written procedures** for addition and subtraction
f drawing, measuring, and estimating in a range of practical contexts

Key Stage 1: Knowledge, Skills, and Understanding

Teaching should ensure that appropriate connections are made between the section *Number and measures* and *Shape, space, and measures.*

Key Stage 1: <u>Ma1:</u> Number and measures

1. *Using and applying 'Number and measures'*
Pupils should:

Problem solving

a explore, interpret, develop flexible approaches to, and persist with problems involving number and measures in a variety of forms

Communicating

a use correct language, symbols, and vocabulary associated with number and measures b explain and record methods and results in spoken, pictorial, and written form

Reasoning

a present results in an organised way; sort and classify numbers according to given criteria

b understand that some statements are **exact** and can be clearly demonstrated

2. *Numbers and the number system*

Pupils should:

Counting

- a count reliably at first up to 20 objects, later extending counting to 100 and beyond (to 120 say), remaining secure across 'tens boundaries' [e.g. from 19 to 20, or from 99 to 100]; recognise the invariance of quantity
- b estimate a number of objects that can be checked by counting; round two digit numbers to the nearest 10

The base 10 number system

- a understand the groupings into units and 10s (and later into 100s) that underpin **place value**; know what each digit represents (including 0 as a number, and as a placeholder), and how the 'value' represented by each digit is determined by its position
- b read and write two-digit and three-digit numbers in figures and words
- c order two-digit numbers and position them on a number line; use $=$, $<$, and $>$

Number patterns and sequences

- a create, describe, and explore basic number patterns and sequences—including odd and even numbers, multiples of 2, multiples of 5, and multiples of 10

3. *Calculation*

Pupils should:

Number operations and the relationship between them

a understand addition and use related vocabulary and notation; understand subtraction (as 'take away' and as 'difference'), and use the related vocabulary and notation; recognise that subtraction is the inverse of addition

b identify and use the calculations needed to solve simple **word problems** and **inverse** problems [e.g. oral "I'm thinking of a number" problems]

c understand simple instances of multiplication as repeated addition, and division (as 'grouping', and as 'sharing'); use the vocabulary and notation associated with multiplication and division; find one half of, or one quarter of a familiar shape, or of a small set of objects

Mental, informal, and standard written methods

a develop instant recall of number facts; know addition and subtraction facts with totals less than 10, and use these to derive other facts; learn addition facts with totals up to 20

b know $\times 2$, $\times 5$, and $\times 10$ multiplication tables, and derive the corresponding division facts; know the doubles of numbers to 20 and the corresponding halves

c use practical and informal written methods to add and subtract two-digit numbers

d develop mental methods which flexibly use known facts to calculate the answer to less familiar 'sums' [e.g. working out 4×6 by doubling 2×6, or by doubling 4×3]; add 10 to any single digit number, then add and subtract a multiple of 10 to or from a two-digit number

e make sense of number sentences involving all four operations

f lay out and complete simple two-digit additions and subtractions in **standard column format**

g use practical and informal written methods and related vocabulary to support multiplication and division, including calculations with remainder

4. *Solving numerical problems*

Pupils should:

a choose sensible calculation methods to solve simple **word problems** involving whole numbers—including problems involving money or measures, drawing on their understanding of arithmetical operations

5. *Processing, representing, and interpreting data*

Pupils should:

a solve suitable problems using simple lists, tables, and charts to sort, classify, and organise information; discuss the methods they use and explain what they find

Key Stage 1: Ma2: Shape, space, and measures

1. *Using and applying 'Shape, space, and measures'*

Pupils should:

Problem solving

a follow instructions to construct simple 2D and 3D objects; represent 3D objects via 2D drawings

Communicating

a use correct language and vocabulary for shape, space, and measures
b measure objects using *ad hoc* informal as well as standard measures; record measurements in ordered tables

Reasoning

a recognise simple spatial patterns and relationships; sort and classify shapes according to given criteria

2. *Understanding properties of shapes, position, and movement*

Pupils should:

a describe relationships using the language "larger – smaller", "higher – lower", "longer – shorter", "above – below", "left of – right of"
b draw and describe properties of 2D and 3D shapes; recognise, name, and sort common 2D and 3D shapes—including triangles, rectangles (including squares), circles, cubes, cuboids, hexagons, pentagons, cylinders, pyramids, cones, and spheres
c recognise right angles; understand whole turns, and quarter- and half-turns (clockwise and anticlockwise)

3. *Understanding measures*

Pupils should:

a use direct comparison to order objects by size, using appropriate language; put familiar events in chronological order

b measure and draw straight lines accurate to the nearest centimetre

c estimate, compare, and measure lengths, weights, and capacities; choose and use standard units (m, cm, kg, litre); compare durations (using seconds, minutes, hours, days); read and interpret numbers on scales to the nearest labelled division, interpreting the divisions between them; identify time intervals, including those that cross the hour

Key Stage 2: Brief version

Key Stage 2: Breadth of Study

During the key stage pupils should be taught the required *Knowledge, Skills, and Understanding* through:

a extending **place value** to larger integers and to simple decimals
b extending their understanding of the number system to include integers, fractions, and decimals
c **learning key facts by heart**; learning to store hundreds, tens, and units temporarily in the mind (including as intermediate outputs in a longer calculation) to support the development of mental calculation strategies
d extending **exact** arithmetic to the **standard written algorithms** for integers and simple decimals
e using **structural arithmetic** to calculate efficiently and to develop (pre-)algebraic thinking
f drawing and measuring; using **exact** arithmetic to make good **estimates** when solving problems; recording results using words, pictures, numbers, diagrams, and tables (and symbols where appropriate)
g linking the language of mathematics with spoken and written English using carefully crafted problems; solving **word problems**; establishing connections between number work, measures, geometry, and practical tasks; distinguishing between sensible and misleading uses of mathematics

Key Stage 2: Knowledge, Skills, and Understanding

Teaching should ensure that appropriate connections are made between the section *Number and measures* and the section *Geometry and measures.*

Key Stage 2: Ma1: Number and measures

1. *Using and applying 'Number and measures'*

Pupils should:

Problem solving

a extract numerical, geometrical, and logical information from simple problems expressed in words
b make **connections**; use integers, decimals, and fractions (and arithmetic) when solving problems involving measures, and in other settings
c solve **multi-step**, and simple **inverse** problems
d solve problems involving tables, lists, and information presented pictorially
e use knowledge of **exact** arithmetic to make good mental **estimates**

Communicating

a use notation, terminology, symbols, and language correctly
b present results and solutions to problems clearly; explain reasoning, methods, and conclusions
c interpret tables, lists, and charts; construct and interpret frequency tables

Reasoning

a present results in an organised way; sort and classify numbers and shapes according to given criteria
b investigate apparent patterns; understand that some statements are **exact** and can be clearly explained

2. *Numbers and the number system*

Pupils should:

Counting

a count reliably beyond 100, passing smoothly from any given set of "90s" onto the next hundred
b count on and back in steps of constant size, starting from any integer, extending to negative integers

The base 10 number system

a use **place value** in representing numbers first up to 1000, then up to 10 000 and beyond; extend to decimals with up to three decimal places

Number patterns and sequences

a recognise two- and three-digit multiples of 2, 5, and 10; find the factors of a given integer, and the common factors of two given integers; recognise prime numbers to 50, and square numbers to 10×10; find factor pairs and all the factors of any two-digit integer; double or halve any two-digit integer

Integers

a read, write (in figures and words), and order whole numbers to 10 000

b multiply, and divide, any integer by 10 or 100, and then by 1000; round integers to the nearest 10 or 100, and then 1000

c understand and use negative integers; order a set of positive and negative integers

Integers and decimals

a use decimal notation for tenths, hundredths, and thousandths; order a set of numbers or measurements

b compare and order integers and decimals in different contexts; locate integers (positive and negative), fractions, and decimals on the number line; use correctly the symbols $=$, $\neq$, $<$, $>$, and also $\leq$, $\geq$

c multiply and divide, any integer or decimal by 10 or 100; round integers and decimals to the nearest integer, to the nearest 'ten', and to the nearest tenth

Fractions, percentages and ratio

a understand fractions; locate fractions on a number line; find fractional parts of shapes or quantities

b understand equivalent fractions; **simplify** by cancelling common factors

c order simple fractions

d understand percentage; use simple percentages for comparison; find fractions and percentages of whole number quantities, and express part of a given whole as a percentage; express one whole number quantity as a fraction of another

e divide a given quantity into two parts in a given **ratio** (both part-to-part and part-to-whole); compare quantities in a given (external) *ratio*; {solve simple problems involving *ratios*}

3. *Calculation*

Pupils should:

Number operations and the relationship between them

a develop their understanding of the four number operations—including inverses, and operations with zero

b find remainders after division; express a quotient as a fraction or decimal; relate $\frac{p}{q}$ to $p \div q$

c know and use the conventions for the order of operations; understand and use **structural arithmetic** to **simplify** calculations; write numerical expressions involving brackets; group related terms in a sum and related factors in a product to *simplify*, and hence evaluate, numerical expressions

Mental methods

a achieve instant recall of all addition and subtraction facts for integers up to 20

b add or subtract any pair of two-digit integers; handle suitable three-digit and four-digit additions and subtractions presented in written form

c add and subtract positive and negative integers mentally

d achieve instant recall of multiplication tables to 10×10 and use them to derive division facts

e multiply and divide in the range 1 to 100, then for larger numbers

f derive multiplication and division facts involving decimals

g relate fractions to multiplication and division; work with simple quotients as fractions and as decimals; add and subtract simple fractions by reducing to a common denominator

Written methods

a use the **standard written method** in column format to add and subtract three-digit positive integers, then four-digit positive integers; add and subtract numbers involving decimals

b use the *standard written method* in column format for short multiplication (of two- and three-digit integers by a single digit multiplier), then long multiplication of two-digit and three-digit integers by two-digit multipliers; extend to simple decimal multiplication
c use short division of two-digit and three-digit integers by a single digit divisor
d use **approximations** and other strategies to check that answers are reasonable

Measures

a calculate reliably with standard measures, money, and time; convert measures from one unit to a related unit
b relate distance, time, and speed in uniform motion

4. *Solving numerical problems*

Pupils should:

a use the four number operations to solve **word problems** involving numbers, money, measures of length, area, mass, capacity, and time
b solve **multi-step** and **inverse** problems with confidence
c check that their results are reasonable; explain why their answers are correct

5. *Processing, representing, and interpreting data*

Pupils should:

a solve suitable problems using simple lists, tables, and charts to sort, classify, and organise information; discuss the methods they use, interpret their results, and explain what they find
b explore the notions of ‘centre’ and ‘spread’ for numerical data sets

Key Stage 2: Ma2: Geometry and measures

1. *Using and applying 'Geometry and measures'*

Pupils should:

Problem solving

a recognise standard geometrical figures; use their properties to select and perform appropriate calculations; measure and draw accurately to construct 2D and 3D figures

b use standard units of measurement; convert reliably between related units

Communicating

a use geometrical notation, terminology, and symbols correctly; interpret solutions to problems involving geometrical figures and measures; organise work and record findings clearly

Reasoning

a analyse standard 2D and 3D figures; calculate efficiently and make simple deductions with angles, lengths, areas, volumes, time, and other measures

2. *Understanding properties of shape*

Pupils should:

a recognise right angles, perpendicular and parallel lines; know that angles at a point total 360°, that angles at a point on a straight line total 180°, and that angles in a triangle total 180°

b describe relationships using the language "larger – smaller", "higher – lower", "longer – shorter", "above – below", "left of – right of", "top – bottom", "in front of – behind", "closer – further away", "between"

c talk clearly about common 2D and 3D shapes; visualise 3D shapes from 2D drawings

d make and draw shapes with increasing accuracy, and analyse their geometrical properties

3. *Understanding properties of position and movement*

Pupils should:

a read and plot coordinates—eventually in all four quadrants; draw, or locate, shapes with given properties in the coordinate plane
b visualise, predict, and represent the position of a shape in 2D following a rotation, reflection, translation, or glide reflection

4. *Understanding measures*

Pupils should:

a draw and measure lines to the nearest millimetre; combine linear measurements to measure perimeters
b draw and measure acute and obtuse angles of a given size to the nearest degree; estimate the size of given angles and order them; draw angles reliably as parts of compound shapes
c read the time to the nearest minute; calculate time intervals from clocks, from timetables, and from calendars
d use standard units of length, area, volume, mass, and capacity; measure and weigh items; convert between related units
e find areas of rectangles and of shapes composed of rectangles; use this to find the exact area of a triangle as 'half base times height'; estimate the area of other 2D shapes
f measure and compare capacities; understand conservation of volume; find volumes of cuboids and of simple shapes composed of cuboids
g read scales with increasing accuracy; record measurements using decimal notation

Key Stage 3: Brief version

Key Stage 3: Breadth of Study

During the key stage pupils should be taught the required *Knowledge, Skills, and Understanding* through:

a extending **place value** to arbitrary integers and decimals
b extending their understanding of numbers to include integers (positive and negative), fractions, and decimals
c extending **exact** arithmetic to the **standard written algorithms** for integers and decimals, and the standard procedures for calculating with fractions
d using **structural arithmetic** for efficient numerical calculation, and for algebraic **simplification** of numerical, fractional, and symbolic expressions
e using their knowledge of *place value* to 'round' numbers and measures, and to make estimates; applying *exact* arithmetic to calculate good *approximations*
f representing unknowns and variables by letters; using formulae; solving linear equations; representing and interpreting straight lines and linear equations
g engaging in tasks that develop short chains of deductive reasoning and that bring out the centrality of **proof** in number, algebra, and geometry
h drawing and measuring; using **ruler and compass constructions**; calculating areas and volumes; recording results using diagrams, words, numbers, and symbols; *angle-chasing* and analysing more complex figures in terms of triangles
i linking the language of mathematics with spoken and written English; building simple logical expressions such as "... and ...", "... or ...", "if ..., then ...", "not only ..., but also ..."
j interpreting carefully crafted **realistic** problems; solving **word problems**; distinguishing between sensible and misleading uses of mathematics
k routinely tackling familiar and unfamiliar problems, including **multi-step** and **inverse** problems; recognising that mathematical operations often come in 'direct-inverse' pairs, and that the inverse operation depends on robust fluency in using the direct operation

l practical work in which they draw inferences from a mathematical analysis of data, and consider how statistics can be used to inform decisions

Key Stage 3: Knowledge, Skills, and Understanding

Teaching should ensure that appropriate connections are made between the section on *Number and algebra* and the section on *Geometry*.

Key Stage 3: Ma1: Number and algebra

1. *Using and applying 'Number and algebra'*

Pupils should:

Problem solving

a use numerical, geometrical, and logical information in analysing data and in solving simple problems

b make connections; use the arithmetic of integers, decimals, and fractions when solving problems

c regularly solve **multi-step** problems and **inverse** problems

d solve problems involving measures, rates and compound measures, ratio and proportion; make and justify **estimates**

Communicating

a use spoken and written language, notation, diagrams, terminology, and symbols correctly

b recognise when information is presented in a misleading way

c present results and solutions to problems clearly, declare unknowns explicitly, and lay out solutions logically **line-by-line**

d interpret tables, lists, and information presented graphically; construct and interpret frequency tables; use precise measures of 'centre' and spread

Reasoning

a understand that some statements can be clearly **proved**, and that other statements can be shown to be false

b use **place value** and **structural arithmetic** to **simplify** calculations and expressions; recognise and use the fact that mathematical operations often come in 'direct-inverse' pairs c use basic results and step-by-step deduction to draw conclusions; investigate apparent patterns and test the validity of statements—proving or disproving these statements conclusively where possible

2. ***Numbers, the number system, structural arithmetic, simplification, and algebra***

Pupils should:

Counting and numbers

a count reliably forwards and backwards across hundreds and thousands boundaries

b solve problems involving **counting** [e.g. How many pages from page 171 to 263?—inclusive and exclusive; How many chords are there joining 10 points on a circle?]

c use **place value** in representing integers to 1 000 000, and decimals with up to four decimal digits; express position as a 'power of 10'; choose the power of 10 to transform a given decimal to an integer (by multiplying)

Sequences; powers and roots

a recognise multiples of 2, 4, 5, 10, 20, 25, 50, 100; factorise instantly any output from multiplication tables to 10×10; recognise (or test quickly) prime numbers to 100 and test possible primes up to 500; recognise square numbers to 20×20; find all the factors of a given integer

b recognise powers of 2, powers of 3, and powers of 5; recognise square and cube roots of familiar squares and cubes; understand and find, or estimate, the square root of any positive number; use index notation for small positive integer powers

c find specified terms of a sequence given a term-to-term or a position-to-term rule; guess the simplest position-to-term rule for the n^{th} term given the first few terms of a sequence

Integers and decimals

a read, write (in figures and words), and order whole numbers and decimals with up to six digits; understand, use, and calculate freely with (positive and negative) integers; use correctly the symbols $=$, $\neq$, $<$, $\leq$, $>$, $\geq$ and the associated language; order a set of positive and negative integers and decimals, or measurements

b use correctly the terms factor, multiple, common factor, common multiple; find and use the *hcf* and *lcm* of two given integers

c multiply, and divide, any integer or decimal by 10, 100, 1000, or 10 000; know the multiplicative complements for 10 (2×5), for 100, and for 1000, and the corresponding decimals [e.g. $\frac{1}{2} = 0.5$, $\frac{1}{5} = 0.2$, $\frac{1}{8} = 0.125$]; recognise as alternative representations the decimal and fraction forms of simple fractions

d express any given large number as a number less than 10 times a power of 10, and a small number as a number greater than or equal to 1 times a power of 10

e compare measurements (in various contexts); round integers and decimals

Fractions, percentages and ratio

a understand general fractions in terms of unit fractions; switch freely between mixed numbers (with fractional part < 1) and standard fractional form $\frac{p}{q}$

b find fractional parts of shapes and quantities, and recognise the fractional part represented; solve simple **ratio** problems

c understand equivalent fractions; express two given fractions with a common denominator; simplify a given fraction; order a list of integers and fractions

d understand 'percentage' as a fractional operator with denominator 100; find fractions and percentages of given quantities; express one quantity as a fraction of another; use the multiplicative character of percentage as an operator in calculations involving percentage increase and percentage decrease; distinguish between absolute and relative increase and decrease

e reduce a *ratio* to its simplest form, and establish the connection with 'fractional parts'; divide a given quantity into two parts in a given *ratio*; solve problems involving **ratio and proportion**

3. *Calculation*

Pupils should:

Number operations and mental methods

a extend existing mental calculation to include negative numbers, decimals, and fractions b calculate effectively in solving problems

Structural arithmetic

a use multiplication tables freely to simplify fractional expressions; convert fractions to decimals, and terminating decimals to their simplified fraction equivalents
b obtain the prime-power factorisation of a given integer by successive division
c understand and use **place value**, inverse operations [e.g. cancellation], and **structural arithmetic** to **simplify** calculations
d understand why $(-1) \times (-1) = 1$; use this to simplify and to evaluate numerical expressions
e use the idea of choosing a suitable (common) denominator to add, subtract, multiply and divide fractions
f solve **word problems** involving rates and **ratios**, including the **unitary method**
g give both roots of simple quadratic equations; simplify numerical expressions involving simple surds [e.g. $\sqrt{8}$ is the same as $2 \times \sqrt{2}$, because both are positive and have the same square]

Algebraic simplification

a substitute numerical values into formulae and expressions; multiply out brackets, collect like terms, identify and take out common factors to simplify expressions; recognise that different-looking expressions may be identical; prove simple algebraic identities, and explain why two given expressions are not identical

Written methods

a relate decimal arithmetic to integer arithmetic; use **standard written methods** in column format for addition and subtraction, short and long multiplication, short {and long} division of integers and decimals

Inequalities

a solve simple linear inequalities in one variable and represent solutions on a number line

Measures

a calculate and work with perimeters, areas, volumes, durations, capacities, and simple compound measures; use standard units of length, area, volume, mass, and capacity; read scales with appropriate rounding; record and order measurements using decimal notation; convert between related units

b estimate the size of any given angle; draw and measure angles reliably to the nearest degree

c calculate reliably with measures; extract and use information from tables and charts; solve **word problems** involving money, time, length, and compound measures (speed, rates)

4. *Algebra: equations, formulae, identities, and functions*

Pupils should:

a set up and solve linear equations in complete generality [e.g. $2 - \frac{3}{4}x = \frac{2-4x}{5}$]; reduce a linear equation in two variables to standard form ($ax + by = c$, or $y = mx + c$); eliminate a variable from two simultaneous linear equations in two unknowns

b change the subject of a formula; draw the graph of a linear function, identifying its gradient, and interpreting its position; construct linear functions arising from real problems, sketch and interpret their graphs; establish the link to **ratio and proportion**

c use letters in general expressions; use index notation for small positive integer powers; **simplify** given expressions

d use algebra to find the exact solution of two simultaneous linear equations in two unknowns by eliminating a variable

e sketch the graphs of simple quadratic functions; solve simple quadratic equations

5. *Solving numerical problems*

Pupils should:

a solve arithmetical problems, **word problems**, and geometry problems involving numbers and measures; check that their results are reasonable

b solve **multi-step** and **inverse** problems with confidence

c use algebraic formulae; set up and solve equations

6. *Processing, representing, and interpreting data*

Pupils should:

a solve problems involving lists, tables, charts, and graphs; sort, classify, and organise information; discuss the methods they use and explain what they find

b find the average (i.e. mean) and other measures of 'centre', and measures of spread for small datasets; identify the modal class for grouped data; interpret frequency diagrams and histograms; use cumulative frequency

c use **counting** where each outcome is 'equally likely' to calculate **probabilities**

Key Stage 3: Ma2: Geometry

1. *Using and applying 'Geometry'*

Pupils should:

Problem solving

a solve geometrical problems involving standard geometrical figures in 2D and 3D, and angles, length, area, and volume

b measure and calculate accurately to construct and analyse 2D and 3D figures; use standard units in geometry

Communicating

a use geometrical language, notation, terminology, and symbols correctly

b work in all four quadrants of the coordinate plane

c lay out calculations, constructions, and **proofs line-by-line**

Reasoning

a use basic geometrical principles to justify each step in a calculation or deduction

b analyse 2D and 3D configurations in terms of triangles

2. *Constructing and analysing geometrical configurations*

Pupils should:

Know and analyse

a recognise right angles, perpendicular and parallel lines, and use the associated language precisely; know that angles at a point total 360°, and that angles at a point on a straight line total 180°

b know that two lines are parallel precisely when alternate angles are equal (or, equivalently, when corresponding angles are equal); prove and use the usual consequences (including the angle sum in a triangle)

c use known angles and angle properties to find unknown angles in given configurations (i.e. *angle-chasing*)

d motivate the formula for the circumference of the circle and estimate π; solve related problems

e talk about and work with common 2D and 3D shapes (including triangles [e.g. right angled, isosceles, and equilateral] and quadrilaterals [e.g. trapezia, parallelograms, rhombuses, rectangles, and squares]); correctly copy drawings from the board; make and draw shapes with increasing accuracy, and analyse their geometrical properties

Constructions and congruence

a use ruler and protractor to draw triangles with given data; extract and apply the basic congruence criteria (SAS, SSS, ASA; RHS) to prove standard results

b draw specified figures using 'ruler' (i.e. *straightedge*) and compasses only; use the basic **ruler and compass constructions** to complete other constructions

Area and Pythagoras

a find the area of rectangles and shapes made from rectangles; find the area of right angled triangles and of general triangles; find the area of a general parallelogram

b find bounds for areas general shapes by counting squares

c relate the formula for the area of a circle to the formula for the circumference; use the formula to solve related problems

d state, prove, and use *Pythagoras' Theorem*

Circles

a understand and use the terms centre, radius, chord, diameter, circumference, tangent, arc, sector, segment

b **prove** the basic properties of a circle [e.g. centre and any chord form an isosceles triangle; angle in a semicircle is a right angle; tangent is perpendicular to radius; tangents from an external point are equal]; apply these results to solve problems

Volume and 3D

a calculate volumes of cuboids and shapes made of cuboids; calculate volumes of a 'wedge' (half a cuboid), polygonal right prisms, and cylinders
b find lengths and angles in simple 3D figures by considering 2D cross-sections

Scaling and enlargement

a draw figures to scale; interpret distances, angles, and areas on maps and other scale drawings

Loci

a interpret a circle as a locus; interpret the perpendicular bisector of a given line segment as a locus

3. *Coordinates and graphs*

Pupils should:

a read and plot coordinates in all four quadrants
b use *Pythagoras' Theorem* to calculate the distance between two given points (simple cases); find the coordinates of the midpoint of a line segment (simple cases)
c establish the link between straight lines in the coordinate plane and linear equations in x and y; understand that parallel lines have the same gradient; find the intersection of two given straight lines
d sketch the graphs of simple quadratic functions
e explore and use coordinates in 3D

Key Stage 4: Brief version

Key Stage 4: Breadth of Study

During the key stage pupils should be taught the required *Knowledge, Skills, and Understanding* through:

a activities that revisit and extend material from Key Stage 3, moving on to achieve **fluency** and **automaticity** in using a wide range of procedures

b using language, terminology, and logic precisely and correctly; linking the language of mathematics with spoken and written English

c learning **basic facts and techniques by heart**; using them to tackle two-step and **multi-step exercises** and **problems** in different contexts, and in solving unfamiliar problems (including **word problems**)

d exploiting **connections** between superficially different topics; **compressing** ideas and techniques

e recognising that operations often come in 'direct-inverse' pairs, that the **inverse** operation is often the more demanding one, and that its mastery depends on robust fluency in the corresponding direct operation

f using **calculators** intelligently where needed, whilst avoiding inappropriate dependence

g extending **exact** arithmetic (without **calculators**) to fractions, surds, and numerical and algebraic expressions involving powers; routinely using algebraic structure to **simplify** numerical, fractional, and symbolic expressions

h making intelligent **estimates** and **approximations** and handling the associated calculations reliably

i combining congruence and **ruler and compass constructions**, parallels, and similarity to establish a **formal** basis for elementary **Euclidean geometry**

j working with tables and information presented graphically; drawing inferences from a mathematical analysis of data drawn from a population with inherent variability; considering how statistics can be used to inform decisions

Key Stage 4: Knowledge, Skills, and Understanding

Teaching should ensure that appropriate connections are made between the section on *Number and algebra* and the section on *Geometry*.

Key Stage 4: <u>Ma1:</u> Number and algebra

1. *Using and applying 'Number and algebra'*

Pupils should:

Problem solving

a use numerical, algebraic, geometrical, and logical information in tackling problems in *Number and algebra*, in solving **word problems**, and in analysing data
b use the structure of arithmetic and the laws of algebra when working with integers, decimals, fractions, surds, and algebraic expressions in solving problems
c regularly solve **multi-step** problems and **inverse** problems
d make use of relevant **connections** between topics
e solve problems involving measures, rates and compound measures, ratio and proportion
f make and justify **estimates**

Communicating

a use spoken and written language, notation, diagrams, terminology, and symbols correctly
b recognise when information is presented in a misleading way
c present results and solutions to problems clearly, declare unknowns explicitly, and lay out solutions and **proofs** logically **line-by-line**
d interpret tables, lists, and charts; present information graphically
e construct and interpret frequency tables; use precise measures of 'centre' and of spread

Reasoning

a investigate apparent patterns; generate, interpret, test, and prove (or disprove) simple conjectures
b use **place value**, index laws, and **structural arithmetic** to **simplify** calculations and expressions, and to justify the exten-

sion of known conventions (including $(-1) \times (-1) = 1$, $2^0 = 1$, $\cos 120° = -\frac{1}{2}$)

c use known results and step-by-step deduction to draw conclusions

2. *From numbers to algebra (including calculation)*

Pupils should:

Numbers and arithmetic

a use **place value** in calculating with decimals; work effectively with very large numbers

b multiply, and divide, any integer or decimal by any power of 10; know the multiplicative complements for powers of 10 [e.g. $1000 = 8 \times 125$], and the corresponding decimals [e.g. $\frac{1}{8} = 0.125$]; recognise the decimal forms of simple fractions

c understand and use **divisibility tests**

d understand why $(-1) \times (-1) = 1$; work with integers, decimals, fractions, and surds—**simplifying** routinely

e solve problems involving **counting**; understand and use the **product rule** for *counting*

f consolidate and extend short and long division

Measures

a compare measurements; round integers and decimals appropriately; use $\approx$ ('approximately equal to') where appropriate

b calculate and work with perimeters, areas, volumes, durations, capacities; use standard units of length, area, volume, mass, capacity, and simple compound measures (speed, density, and other 'rates'); read scales with appropriate rounding; record and order measurements using decimal notation; change between related units—in numerical and algebraic contexts; solve **word problems** involving money, time, length, and compound measures

Bounds and estimation

a understand the limits of accuracy implied by a given measurement in decimal form; interpret the result of an approximate arithmetical calculation; calculate effectively using estimates (without a calculator)

b establish bounds on the accuracy of an **estimate**, and understand how this affects a calculation

Integer factorisation, fractions, and surds

a use the terms factor, multiple, common factor, common multiple; find the *hcf* and *lcm* of given integers
b recognise (or test quickly) prime numbers to 120; use the 'square root test' to identify primes to 1000
c obtain the prime power factorisation of a given integer; list all factors of a given integer
d move freely between "mixed" fractions and fractions in standard fractional form $\frac{p}{q}$; reduce a given fraction to lowest terms; rewrite two given fractions with a common denominator; order a list of fractions
e use factorisation to simplify surd expressions, e.g.

$$\sqrt{12} = \sqrt{2^2 \times 3} = 2\sqrt{3}$$

Fractions

a understand the unit fraction $\frac{1}{q}$ as "that part, of which q identical copies make 1"; understand a general fraction $\frac{p}{q}$ as a multiple $p \times \frac{1}{q}$ of a unit fraction; move freely from a given fraction to a suitable equivalent fraction
b add and subtract fractions; multiply and divide fractions; simplify, and hence evaluate, compound expressions involving fractions

Fractions and decimals

a move freely between terminating decimals and decimal fractions
b know the equivalence of the **exact** (unevaluated) fraction notation
$\frac{p}{q}$ and the result of evaluating $p \div q$; find the decimal of any given fraction; understand why the decimal form of $\frac{p}{q}$ must terminate, or recur
c change any terminating decimal into a fraction in its lowest terms; change any recurring decimal into a fraction

Surds

a recognise $\sqrt{k}$ (for $k > 0$) as the **exact** positive real number whose square is equal to k; given $k > 0$ find the exact or approximate value of $\sqrt{k}$; use the algebra of surds (including rationalising denominators [e.g. $\frac{1}{\sqrt{2}+1} = \sqrt{2} - 1$]); use surds (and π) to calculate *exactly* in geometric contexts; give lengths arising from applications of *Pythagoras' Theorem* and solutions to quadratic equations in *exact* (mixed surd) form

b use the standard notation for, and calculate with, cube roots;

Powers, roots, and the index laws

a factorise instantly any output from multiplication tables to 10×10; recognise square numbers to 25×25; recognise cubes to 6^3

b recognise powers of 2, 3, 4, 5; recognise square and cube roots of familiar squares and cubes; extend powers and roots to simple fractions and decimals; find, or estimate, the square root or cube root of any positive number

c know, understand, and use the index laws; use index notation to present expressions in simplified power form; calculate freely with numerical and algebraic expressions involving powers

d write any given number in standard form; translate a given standard form into the (approximate) number it represents; understand and use 'sig figs'; calculate with numbers given in standard form 'as though they are exact'

e {multiply powers of a fixed base "a" by adding exponents; introduce and use logarithms in *base* 2 and *base* 10}

Fractions, decimals, and percentages

a find and recognise fractional parts of shapes and quantities; express one quantity as a fraction of another

b understand percentage as a fractional operator with denominator 100; know and use the percentage equivalents of familiar fractional parts; work freely with percentages; use the multiplicative character of percentage increase and decrease; solve problems involving percentage change (including **inverse** problems and compound interest)

Sequences

a work with standard integer sequences; generate terms given a term-to-term rule, or a position-to-term rule; guess the simplest position-to-term rule for the n^{th} term given the first few terms of a sequence

b use a given term-to-term rule to find a **closed formula** for the position-to-term rule

c find the term-to-term rule and the position-to-term rule for sequences **defined intrinsically**

d understand that when $x < 1$ $\{$or $|x| < 1\}$ the sequence of powers

(x^n) tends rapidly to 0, and when $x > 1$ {or $|x| > 1$} the sequence of powers (x^n) grows rapidly without bound; link to compound interest, to population growth, to doubling times, and to radioactive half-life

Ratio and proportion

a divide a given quantity into two parts in a given part-to-part, or part-to-whole **ratio**; express the division of a quantity into two parts as a *ratio*; work with separate quantities in a given (external) *ratio*; reduce a *ratio* to its simplest form
b calculate the result of a change of units; draw and use scale diagrams and maps; understand the effect of scaling and enlargement on different quantities (including angles, lengths, areas, and volumes)
c solve **proportion** problems (three out of four quantities being given, determine the fourth); use the **unitary method**, and then the general method, to solve *proportion* problems
d understand and use "X is inversely proportional to Y" as meaning "X is proportional to $\frac{1}{Y}$"

Algebraic expressions

a substitute numerical values into **formulae** and **expressions**
b multiply out brackets, collect like terms, and take out common factors to **simplify** linear, quadratic, and higher order expressions; *simplify* general *expressions* (possibly involving powers and roots) by using additive simplification, the distributive law, and cancellation—giving answers in factorised form; work with algebraic fractions having linear and quadratic denominators
c rearrange *formulae*; solve problems using standard *formulae*; find x given particular values of y in simple equations [e.g. $y = \frac{k}{x}$, or $y = \frac{k}{x^2}$]
d set up linear equations; solve the general linear equation in one unknown
e set up linear equations in two unknowns; interpret a linear equation in two unknowns in the coordinate plane as representing a straight line; draw the graph of a linear **function**, identifying its gradient, and interpreting its position; find the gradient from an equation given in any form; transform a given equation into the form $y = mx + c$ (or $x = a$); construct linear functions arising

from real problems, sketch and interpret their graphs; establish the link to **ratio** and **proportion**

f solve any pair of simultaneous linear equations by eliminating a variable; interpret the analytic solution as 'finding the point of intersection' (if any) of the two lines

g factorise quadratic expressions in one variable; solve quadratic equations by factorising; interpret solutions as those points where the graph crosses the x-axis; solve fractional equations that reduce to quadratics

h factorise and use the difference of two squares; conclude that, if $k > 0$, the equation $x^2 = k$ has two solutions ($\pm\sqrt{k}$); interpret this as a statement about the graph of $y = x^2 - k$

i know and use the expansion of $(x + a)^2$; {extend to $(x + a)^3$;} use this to 'complete the square' for any given quadratic; obtain the formula for the solutions of the general quadratic

$$y = ax^2 + bx + c;$$

use this formula to solve quadratic equations; deduce the symmetry of the graphs of quadratic functions

j solve two simultaneous equations where one is linear and the other quadratic; {find the points where two circles intersect}

k understand the difference between an equation and an **identity**; decide whether two given expressions are identical or not—then **prove** they are, or show that they are not

l solve linear inequalities in one and two variables; interpret the solution graphically

3. ***Coordinates, graphs, and functions***

Pupils should:

a read and plot coordinates in all four quadrants; move freely between straight lines in the coordinate plane and linear equations in x and y; derive the equation of a line through two given points, and the equation of a line through a given point with a given gradient

b find the coordinates of the midpoint of a line segment; calculate the distance between two points in 2D {or 3D}

c interpret straight line graphs arising in real situations; find the gradient and intercept of a given straight line graph

d know and use the general form $y = mx+c$ (or $x = a$) for a straight line; use gradient and intercept; find the point of intersection of two given straight lines

e know that parallel lines have the same gradient; prove and use the fact that two lines with gradients m and m' are perpendicular precisely when $m \cdot m' = -1$

f for particular values of m and c interpret the standard form

$$y = mx + c$$

as '$Y = mX$' relative to an origin at $(0, c)$ or at $(-\frac{c}{m}, 0)$

g sketch the graph of any given quadratic function by completing the square

h sketch other graphs—including simple cubic functions, the reciprocal function $y = \frac{1}{x}$, the exponential function $y = k^x$ for easy (positive) values of k, the circular functions $y = \sin x$, $y = \cos x$ $\{$and $y = \tan x\}$

i use coordinates to solve simple problems in 3D

4. ***Processing, representing, and interpreting data***

Pupils should:

a engage in practical and theoretical work to construct and interpret frequency tables, lists, and information presented graphically; use precise measures of 'centre' and spread; sort, classify, and organise information

b discuss variability; distinguish between data representing a single idealised measure and informal 'random variables' **sampled** from a **population** or distribution

c calculate the mean of a set of numbers or measures; find the median and mode of a given set of numbers; use mode or median as appropriate to summarise the 'centre'; identify the modal class for grouped data; refine measures of spread and 'central tendency'

d introduce ideas of **probability** via standard examples of discrete sample spaces in which each outcome is equally likely; explore the general notion of an *event*

e understand why $\text{prob}(A \cup B) = \text{prob}(A) + \text{prob}(B)$ for disjoint events A, B; use simple counting to calculate probabilities in discrete sample spaces; understand and use the inclusion/exclusion formula

$$\text{prob}(A \cup B) = \text{prob}(A) + \text{prob}(B) - \text{prob}(A \cap B)$$

for events which are not necessarily disjoint

Key Stage 4: Ma2: Geometry

1. *Using and applying 'Geometry'*

Pupils should:

Problem solving

a know and understand basic **ruler and compass constructions**; use these to devise simple constructions

b use standard units in geometry; solve geometrical problems in 2D and 3D involving calculation, construction, and deduction

c measure and calculate accurately to construct and analyse 2D and 3D figures in terms of triangles; use known results to construct simple **proofs**

Communicating

a use geometrical language, notation, terminology, and symbols correctly

b work in all four quadrants of the coordinate plane; interpret a given equation as the graph of a function or a circle

c lay out calculations, constructions, and **proofs line-by-line**

Reasoning

a use the basic principles of **Euclidean geometry** and results derived from them to justify each step in a calculation, construction, or deduction

b analyse 2D and 3D configurations [e.g. by singling out, and using known properties of triangles]

2. *From naïve construction to Euclidean geometry*

Pupils should:

Ruler and compass constructions revisited and organised

a know that two given points A, B determine a line AB, a line segment $\underline{AB}$, and a circle with centre A and radius $\underline{AB}$; relate this to ideal **ruler and compass constructions**

b know and use the conventional notation for labelling the angles and sides of $\triangle ABC$

c accept and use the SAS, SSS, ASA (and later RHS) congruence criteria; prove the basic properties of isosceles triangles; justify the basic *ruler and compass constructions* and use them to devise other constructions

d prove that the perpendicular bisector of a given line segment $\underline{BC}$ is the locus of points X 'equidistant from' B and C; construct the circumcentre of any triangle

e recognise the 'perpendicular distance' from a point X to a line as the (shortest) distance to the line; prove the angle bisector of $\angle BAC$ is the (part-)locus of points equidistant from the lines AB and AC; {construct the incentre of $\triangle ABC$}

f {prove that the three altitudes of a triangle are concurrent}

Parallel lines and angles in a triangle

a know that angles at a point total 360°, and that angles at a point on a straight line total 180°; conclude that 'vertically opposite angles are equal'

b recognise that 'two lines are parallel precisely when alternate angles (or equivalently, when corresponding angles) created by any transversal are equal'; derive the basic properties of a parallelogram and of a rhombus; know and use {and prove} the *Midpoint Theorem*

c prove that the angles in any triangle add to 180° and that the exterior angle at any vertex is equal to the sum of the two interior opposite angles; deduce that the angles in any quadrilateral add to 360°; {calculate the angle-sum in an n-gon, and the angle size in a regular n-gon}

d combine known results about angles to find unknown angles, and to show that certain pairs of lines are parallel

e know and use the fact that the tangent and radius at a point on a circle are perpendicular; conclude that tangents from an external point are equal; prove that the angle subtended by a chord on the major arc is half the angle subtended at the centre O; conclude that angles subtended in the same segment are equal, and that opposite angles of a cyclic quadrilateral add to 180°; prove and use the *Alternate Segment Theorem*

f prove that the area of a parallelogram is equal to that of a rectangle on the same base and between the same parallels and deduce the formula for the area of a triangle; use this to prove *Pythagoras' Theorem*

Similarity

a establish and use the AAA-similarity criterion for general triangles; prove basic results using similarity

b extend the *Midpoint Theorem* to divide a given segment into any number of equal parts; prove and use the *Intercept Theorem*

3. Geometric calculation

Pupils should:

Trigonometry

a show that the standard trig ratios for acute angles θ depend only on the angle θ; understand that $\sin\theta$, $\cos\theta$ take values between 0 and 1, and that $\tan\theta$ can take any positive value

b find the **exact** values of $\sin\theta$, $\cos\theta$, $\tan\theta$ for $\theta = 0°$, $30°$, $45°$, $60°$; plot graphs of $y = \sin\theta$, $y = \cos\theta$ $\{$and $y = \tan\theta\}$ for $0° \leq \theta < 90°$; understand why $\cos\theta = \sin(90° - \theta)$

c use a calculator to find $\arcsin(x)$, $\arccos(x)$, $\arctan(x)$ for given values of x

d calculate missing lengths and angles in a given triangle $\triangle ABC$

e given $\triangle ABC$, derive and use the formula 'area$(ABC) = \frac{1}{2} \cdot ab \cdot \sin C$'; deduce the *Sine Rule* and use it to 'solve triangles'; prove that $\frac{a}{\sin A} = 2R$, where R is the circumradius of triangle ABC

f show that on the unit circle with centre at the origin O, the point P for which the radius OP makes an angle θ with the positive x-axis has coordinates $(\cos\theta, \sin\theta)$; apply *Pythagoras' Theorem* to derive the identity $\sin^2\theta + \cos^2\theta = 1$; use this identity to find values of $\cos\theta$ given the value of $\sin\theta$ (and *vice versa*) $\{$and the value of $\tan\theta$ given the value of $\cos\theta\}$

g prove the *Cosine Rule*, and use it to find unknown lengths and angles in triangles and other 2D and 3D figures

h extend the definition of $\sin\theta$ and $\cos\theta$ to $\theta > 90°$; extend the graphs of the graphs of $y = \sin\theta$, $y = \cos\theta$ to $0° < \theta < 180°$ $\{$and beyond$\}$

i show that in the 'ambiguous (ASS) case', the data may determine two possible triangles

2D and 3D figures

a work freely with standard 2D figures

b draw figures to scale; interpret maps and other scale drawings;

apply similarity in analysing problems; understand how enlargement and scaling (or similarity) affects angles, lengths, areas, and volumes

c find lengths and angles in 3D figures by considering 2D cross-sections; calculate the angle between two planes

d calculate surface areas and volumes of standard figures

Circles

a understand and use the terms centre, radius, chord, diameter, circumference, tangent, arc, sector, segment

b understand and use the formula for the circumference of a circle; calculate the length of circular arcs

c relate the formula for the area of a circle to the formula for the circumference; calculate the area of a sector

d calculate the circumradius {and inradius} of a triangle

e use *Pythagoras' Theorem* to find the equation of a circle of radius r centred at the origin {and at the point (c, d); complete the square to identify quadratic equations as circles in simple cases, and so to find the centre and radius}

f {find the equation of the tangent to a given circle at a specified point}

FULLER VERSIONS

Key Stage 1: Fuller version

Key Stage 1: Breadth of study

During the key stage pupils should be taught the required *Knowledge, Skills, and Understanding* through:

a practical activity, exploration and discussion—especially to reinforce key ideas such as **place value**, and the link between number and measures

b linking the language of mathematics with spoken and written English through class discussion (taller/shorter, less than/more than, left/right, between, above/below, in front of/behind, etc.) and carefully crafted **realistic** problems (including **word problems**)

c using mathematical ideas in practical activities, then recording these using objects, pictures, diagrams, tables, words, and numbers

d **learning key facts by heart**; learning to store tens and units temporarily in the mind as intermediate outputs in a longer calculation—to support the development of mental calculation strategies

e drawing, measuring, and estimating in a range of practical contexts

f exploring and using a variety of resources and materials

g activities that encourage them to make **connections** between basic number work and other aspects of their work in mathematics and elsewhere

Key Stage 1: Knowledge, Skills, and Understanding

Teaching should ensure that appropriate connections are made between *Number and measures* and *Shape, space, and measures*.

Key Stage 1: Ma1: Number and measures

1. *Using and applying 'Number and measures'*

Pupils should be taught to:

Problem solving

a consider, interpret, and explore problems involving number, measures, and data presented in a variety of forms; identify what they need to do to solve a given problem
b develop flexible approaches to problem solving and look for ways to overcome difficulties
c make considered decisions about which operations and strategies to use
d organise and check their work

Communicating

a use correct language, symbols, and vocabulary associated with number, measures, and data
b present results and solutions to puzzles and problems in an organised way; explain decisions, methods, and results in spoken, pictorial, and written form, at first using informal language and recording, then correct mathematical language and symbols

Reasoning

a understand a general statement [e.g. "A multiple of 5 always ends in 5"] and investigate whether particular cases confirm it; sort and classify numbers according to given criteria
b understand that some statements are **exact**, or correct, and can be clearly explained or demonstrated
c explain their methods and reasoning when solving problems involving number and measures

2. *Numbers and the number system*

Pupils should be taught to:

Counting

a count reliably—at first up to 20 objects; be familiar with the numbers 11 to 20; gradually extend counting to 100 and beyond (to 120, say), ensuring that the sequence is secure across

'9s boundaries'; count on and back in steps of 1, 2, 5, 10; recognise that if the objects are rearranged the total number stays the same

b estimate a number of objects that can be checked by counting; round two-digit numbers to the nearest 10

The base 10 number system

a use action and talk to understand the groupings into 1s and 10s (and later 100s) that constitute **place value** in representing numbers up to 100 (and beyond); know what each individual digit represents (including the meaning of 0 as a number and as a place-holder), and how the 'value' of a digit is determined by its position

b read and write two-digit and three-digit numbers in figures and words, first to 20, then to 100 and beyond; partition two-digit numbers in different ways

c understand and use the vocabulary of comparing and ordering familiar numbers; order two-digit numbers and position them on a number line; use the = sign, and begin to use $<$ and $>$ signs and the associated language

Number patterns and sequences

a create and describe number patterns; explore and record patterns related to addition and subtraction; explore and record multiples of 2, 5, and 10, explaining the evident patterns and using them to make predictions

b recognise simple sequences, including odd and even numbers, first to 30—then beyond

3. *Calculation*

Pupils should be taught to:

Number operations and the relationship between them

a understand addition and use the vocabulary and notation related to addition; recognise that summands can be combined in any order

b understand subtraction in both forms (as 'take away' and as 'difference') and use the related vocabulary and notation; recognise that subtraction is the inverse of addition; give the subtraction corresponding to an addition and vice-versa; relate addition and subtraction to 'counting on'; use the symbol = to represent equality; solve simple missing number problems [e.g. $6 = 2 + \ldots$]

c identify the calculations needed to solve given **word problems** and **inverse** problems [e.g. "I'm thinking of a number" problems]; invent appropriate *word problems* which embody given calculations

d understand (through action and talk) simple instances of multiplication as replication or repeated addition [e.g. 3 groups with 5 sweets in each group; counting dots in rectangular arrays]; interpret and solve related *word problems*

e explore doubling and halving; recognise that each is the inverse of the other

f use the language of halves and quarters; find one half or one quarter of a familiar shape, or of a given small set of objects; begin to understand division as 'grouping' and as 'sharing'; use vocabulary associated with multiplication and division

Mental, informal, and standard written methods

a develop instant recall of number facts; know addition and subtraction facts with totals less than 10; then use these facts and **place value** to derive and learn facts with totals to 20, and all pairs of multiples of 10 with totals up to 100

b know multiplication facts for the $\times 2$, $\times 5$, and $\times 10$ multiplication tables, and derive the corresponding division facts; know the doubles of numbers to 20 and the corresponding halves

c add and subtract mentally a one-digit number or multiple of 10 from any two-digit number; use practical and informal methods [e.g. partitioning to create easy intermediate steps] to add and subtract two-digit numbers

d develop mental methods for using known facts to calculate the answer to less familiar 'sums', including adding 10 to any given single digit number, and subtracting a small multiple of 10 from a given two-digit number; develop methods for adding and subtracting, including making use of the fact that summands can be added in any order and that subtraction is the inverse of addition

e carry out simple calculations [e.g. $40 + 20 = \ldots$, $40 + \cdots = 100$, $56 - \cdots = 10$, $\ldots \times 4 = 8$, $\ldots \div 2 = 6$, $30 - \ldots = 24$]; record calculations and the results of calculations in a 'number sentence', using the symbols $+$, $-$, $\times$, $\div$ and $=$ correctly [e.g. $7 + 2 = 9$]; interpret number sentences involving all four operations f lay out and complete simple one-digit and two-digit additions and subtractions in **standard column format**

g represent repeated addition, and the number of objects in a rectangular array, as multiplication; represent sharing and repeated subtraction as division; use practical and informal written methods and related vocabulary to support multiplication and division, including calculations with remainder

4. ***Solving numerical problems***

Pupils should be taught to:

a solve simple **inverse** problems [e.g. "I'm thinking of a number" problems and "missing digit" problems]

b choose sensible calculation methods to solve **word problems** involving whole numbers (including problems involving money or measures), drawing on their understanding of arithmetical operations

c check that their answers are reasonable; describe their methods and reasoning, and explain why their answers are correct

5. ***Processing, representing and interpreting data***

Pupils should be taught to:

a solve suitable problems by using simple lists, tables, and charts to sort, classify, and organise information, discuss the methods they use and explain what they find

Key Stage 1: Ma2: Shape, space, and measures

1. ***Using and applying 'Shape, space, and measures'***

Pupils should be taught to:

Problem solving

a measure objects using informal and standard measures; record findings; select and use appropriate equipment and materials when solving problems involving measures or measurement

b try different approaches and find ways of overcoming difficulties when solving problems involving shape, space, and measures

Communicating

a use correct language and vocabulary for shape, space, and measures

b record measurements in ordered tables

c follow instructions to construct simple 2D and 3D objects; represent 3D objects in 2D drawings

Reasoning

a recognise simple spatial patterns and relationships [e.g. that the number of corners and the number of edges of a polygon are equal] and make predictions; test these predictions and discuss what they find

b sort and classify shapes according to given criteria; use mathematical language to communicate and explain

2. *Understanding properties of shapes, position and movement*

Pupils should be taught to:

a describe relationships using the language “larger – smaller”, “higher – lower”, “longer – shorter”, “above – below”, “left of – right of”

b draw and describe properties of 2D and 3D shapes that they can see or visualise; use the related vocabulary; identify common shapes from pictures of objects in different orientations

c observe, handle, and describe common 2D and 3D shapes; sort, name, and make common 2D and 3D shapes, and describe their mathematical features—including triangles of various kinds, rectangles (including squares), circles, cubes, cuboids, then hexagons, pentagons, cylinders, pyramids, cones, and spheres

d observe, visualise, and describe positions, directions and movements using common words

e recognise and use accurate vocabulary to describe movements in a straight line, or involving rotations, and combine them in simple ways; follow and give instructions [e.g. to get from the classroom to the headteacher’s office; or to rotate a programmable toy]

f recognise right angles in different orientations; work with quarter- and half-turns (clockwise and anticlockwise)

3. *Understanding measures*

Pupils should be taught to:

a use direct comparison to estimate the size of objects and to order them by size, using appropriate language; put familiar events in chronological order

b use a ruler to draw and measure lines to the nearest centimetre

c compare and measure objects first using uniform non-standard units [e.g. using a straw, or identical wooden cubes to 'measure' length], then with a standard unit of length (cm, m), of weight (g, kg), or of capacity (ml, l) [e.g. "longer or shorter than a metre rule"; "three and a bit litre jugs"]; compare duration using standard units of time (seconds, minutes, hours, days) and know the relationship between them

d understand the importance of standard units; choose and use appropriate standard units and simple measuring instruments; read and interpret numbers on scales to the nearest labelled division, then interpret the divisions between labelled marks

e read the time to a quarter of an hour; identify time intervals, including those that cross the hour

Key Stage 2: Fuller version

Key Stage 2: Breadth of Study

During the key stage pupils should be taught the required *Knowledge, Skills, and Understanding* through:

- a extending their grasp of **place value** to include larger integers and simple decimals
- b activities that extend their understanding of the number system to include integers, fractions, and decimals
- c extending **exact** arithmetic to include the **standard written algorithms** (in column format) for integers and for simple decimals
- d **learning key facts by heart**; learning to store hundreds, tens, and units temporarily in the mind [e.g. as intermediate outputs in a longer calculation] to support the development of mental calculation strategies
- e using **structural arithmetic** to calculate efficiently and to develop (pre-)algebraic thinking
- f drawing and measuring in a range of contexts; using their grasp of *exact* arithmetic to make suitable **estimates** and **approximations** when solving problems; recording results using pictures, diagrams, words, and numbers (and symbols where appropriate)
- g linking the language of mathematics with spoken and written English (taller/shorter, less than/more than, left/right, between, above/below, in front of/behind, horizontal/vertical, etc.); using carefully crafted **realistic** problems; solving **word problems**; establishing **connections** between number work, measures, geometry, and practical tasks; distinguishing between sensible and misleading uses of mathematics

Key Stage 2: Knowledge, Skills, and Understanding

Teaching should ensure that appropriate connections are made between the section *Number and measures* and the section *Geometry and measures*.

Key Stage 2: Ma1: Number and measures

1. *Using and applying 'Number and measures'*

Pupils should be taught to:

Problem solving

a extract numerical, geometrical, and logical information from simple problems expressed in words and identify the appropriate calculations needed to solve the problem

b make **connections** and use integers, decimals, and fractions (and arithmetic) flexibly and reliably when solving problems involving measures, and in other parts of the mathematics curriculum

c solve **multi-step** and simple **inverse** problems; break down a more complex problem or calculation into simpler steps, identify the information needed to carry out each step, and complete the sequence of steps efficiently; when stuck, look for an alternative approach which gets round the difficulty

d solve problems involving tables and lists, or other pictorially presented information

e use their grasp of **exact** arithmetic to make mental **estimates** of the answers to calculations

Communicating

a use language, terminology, diagrams, notation, and symbols correctly within a given problem; recognise when information is presented in a misleading way

b present results and solutions to puzzles and problems in an organised way; interpret solutions in the context of the problem; explain reasoning, methods, and conclusions in spoken, pictorial, and written form, using mathematical language and symbols correctly; organise and check their work

c interpret tables, lists, and charts in everyday life; construct and interpret frequency tables, including tables for grouped data; represent and interpret data; refine ways of recording

Reasoning

a present results in an organised way; sort and classify numbers and shapes according to given criteria

b develop logical thinking; search for patterns, investigate apparent patterns, and test the validity of statements— proving or disproving statements conclusively where possible [e.g. “there are exactly four prime numbers less than 10”; “ a square number can never end in 2”; “the digits of any multiple of 9 add to 9”; “the digits of any multiple of 9 add to a multiple of 9”]

c understand that some statements are **exact** or correct—and can be clearly explained or demonstrated, that other statements may be false—and can be shown to be false (and that the validity of some statements may have to be left unresolved for the time being)

d explain their methods and reasoning when solving problems involving number, data, and measures

2. *Numbers and the number system*

Pupils should be taught to:

Counting

a count beyond 100, performing reliably when crossing from any given set of “90s” to the next hundred

b count on and back in tens or hundreds from any two-digit or three-digit starting number; recognise and continue number sequences formed by counting on or back in steps of constant size from any integer, extending to negative integers when counting back

The base 10 number system

a extend their understanding and use of **place value** in representing numbers first up to 1000, then up to 10 000 and beyond; extend *place value* to decimals with up to three decimal places

Number patterns and sequences

a recognise and describe two- and three-digit multiples of 2, 5, and 10; observe and explain more general patterns and use these to make predictions; make general statements using words to describe an apparent relationship, and then test it; find factors of a given integer, and the common factors of two (or more) given integers; recognise prime numbers to 20, then to 50 and beyond, and square numbers up to 10×10; find factor pairs and all the factors of any two-digit integer b explore variations on doubling and halving [e.g. tripling and "thirding"]

Integers

a read, write (in figures and words), and order whole numbers first to 1000, then to 10 000, recognising that the position of a digit determines its 'value'

b round integers to the nearest 10 or 100, and then 1000; give estimates for sums and differences; multiply and divide any integer by 10 or 100, and then by 1000

c understand and use negative integers, position them on a number line, and use the relevant language correctly; order a set of positive and negative integers, explaining methods and reasoning

Integers and decimals

a understand and use decimal notation for tenths, hundredths, and later thousandths [e.g. order amounts of money; round a sum of money to the nearest £; convert a length such as 1.36 metres to centimetres, or 568 ml to litres]; locate, and hence order, a set of numbers or measurements on a number line

b compare and order, a set of integers (positive and negative), fractions, and decimals, or measurements, in different contexts; use correctly the symbols $=$, $\neq$, $<$, $>$ (and also $\leq$, $\geq$) and the associated language; relate inequality [e.g. $-3 < -1$] to relative position on the number line

c multiply and divide, any integer or decimal by 10 or 100; round decimals with up to three decimal places to the nearest integer, to the nearest 'ten', and to the nearest tenth

d recognise as alternative representations the decimal and fractional forms of halves, quarters, tenths, and hundredths, and other familiar fractions (including fifths and thirds)

Fractions, percentages and ratio

a understand unit fractions [e.g. $\frac{1}{3}$ or $\frac{1}{8}$], then fractions obtained by replicating unit fractions [e.g. $\frac{2}{3} = 2 \times \frac{1}{3}$ or $\frac{5}{8} = 5 \times \frac{1}{8}$]; use the language of numerator and denominator correctly; locate fractions on a number line

b identify illustrated fractional parts of shapes or quantities

c understand equivalent fractions; **simplify** fractions by cancelling common factors in numerator and denominator

d compare and order simple fractions [e.g. with denominators 2, 4, 5, 10] by converting them to fractions with a common denominator; explain their methods and reasoning

e understand that 'percentage' represents an equivalent fraction with denominator 100; interpret this as "number of parts per 100"; use percentages for comparison; find fractions and percentages of whole number quantities [e.g. $\frac{5}{8}$ of 96, 65% of £260]

f recognise parts of a whole and use simple fractions and percentages to describe them; express a given part of a given whole as a percentage [e.g. express £400 as a percentage of £2000] and as a fraction; express one whole number quantity as a fraction of another [e.g. recognise that 8 slices of a 5-slice pizza represents $\frac{8}{5}$ or $1\frac{3}{5}$ pizzas]; explain their methods and reasoning

g use **ratio** notation; reduce a *ratio* to its simplest form; divide a given quantity into two parts in a given *ratio* (both part-to-part and part-to-whole); compare quantities in a given (external) *ratio*; {solve simple problems involving *ratios*}

3. *Calculation*

Pupils should be taught to:

Number operations and the relationship between them

a develop further their understanding of the four number operations and the relationships between them, including inverses; handle addition and subtraction of the number 0, and multiplication by 0; know that division by 0 is impossible; use the related vocabulary; choose suitable number operations to solve a given problem

b find remainders after division; express a quotient as a fraction or decimal; establish the link between $\frac{p}{q}$ and $p \div q$; round answers up or down after division as is appropriate in the context

c understand that brackets determine the order of operations; evaluate numerical expressions with and without brackets; group related terms in a sum and related factors in a product to **simplify** and hence evaluate numerical expressions; understand and use **structural arithmetic**— based on **place value**, and the commutative, associative and distributive laws—to simplify expressions and calculations involving first addition and multiplication, then subtraction and division [e.g. $148 + 29 + 52 = \ldots$, $6 \times 23 + 14 \times 23 = \ldots$, $6 \times 23 + 7 \times 46 = \ldots$, $4 \times 23 \times 25 = \ldots$; $148 + 76 - 48 = \ldots$, $148 + 89 - (60 - 52) = \ldots$, $54 \times (35 \div 27) = \ldots$]; use this to carry out mental and written calculations efficiently

Mental methods

a practise to achieve instant recall of all addition and subtraction facts for integers up to 20

b work out the additive supplement needed to get from any given two-digit number to 100; add or subtract any pair of two-digit integers; handle suitable three-digit and four-digit additions and subtractions presented in written form [e.g. using compensation and other methods to work out $3000 - 1997$, $4560 + 998$]

c add and subtract positive and negative integers mentally

d practise to achieve instant recall of multiplication tables to 10×10 and use them to derive quickly the corresponding division facts

e multiply and divide, at first in the range 1 to 100 [e.g. 27×3, $65 \div 5$], then for particular cases of larger numbers by using factors, distributivity, **structural arithmetic** and other methods [e.g. $10 \times 27 = \ldots$, $5 \times 26 = 5 \times (2 \times 13) = 10 \times 13 = \ldots$, $3 \times 17 + 7 \times 17 = \ldots$]

f use place value and multiplication tables to derive multiplication and division facts involving decimals [e.g. $0.8 \times 7 = \ldots$, $4.8 \div 6 = \ldots$]

g relate fractions to multiplication and division [e.g. $6 \div 2 =$ '$\frac{1}{2}$ of 6' $= 6 \times \frac{1}{2}$]; express a simple quotient as a fraction and as a decimal [e.g. $67 \div 5 = 13\frac{2}{5} = 13.4$]

h add and subtract simple fractions by reducing to a common denominator

Written methods

a use **standard written method** in column format to add and subtract three-digit positive integers (< 1000), then four-digit positive integers ($< 10\ 000$); then add and subtract numbers involving decimals; use approximations and other methods to check that answers are reasonable

b use *standard written method* in column format for short multiplication (of two- and three-digit integers by a single digit multiplier), then long multiplication of two-digit and three-digit integers by two-digit multipliers; extend to simple decimal multiplication

c use *standard written method* for short division of two-digit and three-digit integers by a single digit divisor; extend division to informal methods of dividing by a two-digit divisor [e.g. $64 \div 16$, or $176 \div 16$]

d use **approximations** and other strategies to check that answers are reasonable

Measures

a calculate reliably with standard measures, money, and time
b convert measures from one unit to a related unit; convert between centimetres and millimetres or metres, then between millimetres and metres, and between metres and kilometres, between grams and kilograms; explain their methods and reasoning
c relate distance, time, and speed in uniform motion

4. *Solving numerical problems*

Pupils should be taught to:

a choose, use and combine any of the four number operations to solve **word problems** involving numbers— including numbers in realistic settings—as money, or measures of length, mass, capacity, or time, then perimeter and area
b choose and use an appropriate calculation method when faced with a problem or puzzle, and explain their methods and reasoning; solve sharing and grouping problems that require rounding down [e.g. 'How many sweets each when 100 sweets are shared between 7 children?'] and rounding up [e.g. How many minibuses are needed to transport 100 children if each minibus carries 13 children?']; solve simple **multi-step** and **inverse** problems with confidence
c identify the calculations relevant to given *word problems*; invent appropriate *word problems* which embody given calculations
d estimate answers by approximating; check that their results are reasonable by thinking about the context of the problem, and where necessary by checking accuracy; explain why their answers are correct
e recognise, represent, and interpret simple number relationships; construct and use formulae in words, and occasionally in symbols [e.g. $A = l \times b$ gives the area in cm^2 of a rectangle of length $l$cm and breadth $b$cm; $c = 15n$ is the cost in pence of n articles at 15p each]

5. *Processing, representing and interpreting data*

Pupils should be taught to:

a solve suitable problems by using simple lists, tables, and charts to sort, classify, and organise information; read and present information in pictograms, bar charts, pie charts, and line graphs; discuss the methods they use, interpret their results, and explain what they find

b explore the notions of 'centre' and spread for numerical data sets

Key Stage 2: Ma2: Geometry and measures

1. *Using and applying 'Geometry and measures'*

Pupils should be taught to:

Problem solving

a recognise rectangles, triangles, cubes, cuboids, etc. in tackling geometrical problems; use their properties accordingly; select and use appropriate calculations

b measure and draw accurately to construct 2D and 3D figures

c approach spatial problems flexibly, including trying alternative approaches to overcome difficulties; use checking procedures to confirm that their results of geometrical problems are reasonable

d recognise the need for standard units of measurement, and use them appropriately, converting reliably between related units

Communicating

a use geometrical language, notation, and symbols correctly

b present and interpret solutions to problems involving geometrical figures and measures; organise work and record results clearly

Reasoning

a use mathematical reasoning to analyse standard 2D and 3D figures (rectangles, triangles, cuboids, etc.); explain features of 2D and 3D figures

b calculate efficiently and make simple deductions with angles, lengths, areas, volumes, time, money, and other measures

2. *Understanding properties of shape*

Pupils should be taught to:

a recognise right angles, perpendicular and parallel lines in 2D and 3D figures; know that angles are measured in degrees, that one whole turn is 360°, and that angles at a point total 360°; recognise that angles at a point on a straight line total 180°; know that the sum of the angles in a triangle is 180°

b describe spatial relationships and compare objects using the language "larger – smaller", "higher – lower", "longer – shorter", "above – below", "left of – right of", "top – bottom", "in front of – behind", "closer – further away", "between"

c talk clearly about common 2D and 3D shapes; sharpen their geometrical language, especially relating to squares and general rectangles, triangles, and cuboids, and extending to quadrilaterals, prisms, and pyramids of various kinds; recognise that shapes in different positions may be identical; visualise common 3D shapes from 2D drawings

d make and draw standard configurations [e.g. right angles, parallel lines] and shapes [e.g. equilateral triangles, rectangles, regular hexagons, circles] with increasing accuracy; construct, and analyse the geometrical properties of, equilateral triangles, squares, rectangles, regular hexagons, and circles

3. *Understanding properties of position and movement*

Pupils should be taught to:

a read and plot coordinates—initially in the first quadrant only (later extending to all four quadrants); draw, or locate, shapes with given properties in the coordinate plane; complete specified shapes given partial information [e.g. plot all four vertices of a rectangle given the coordinates of three of the four vertices]

b visualise and describe position and movement using appropriate language; visualise, predict and represent the position of a shape following a rotation, reflection, translation, or glide reflection

4. *Understanding measures*

Pupils should be taught to:

a draw and measure lines to the nearest millimetre; combine linear measurements to measure perimeters; measure the perimeter of simple rectilinear 2D figures; draw specified 2D shapes [e.g. rectangles of given dimensions]; understand that different shapes, and shapes of different areas can have the same perimeter

b draw and measure given acute and obtuse angles to the nearest degree (ultimately using a protractor); estimate the size of angles; recognise angles as greater or less than a right angle or half-turn; estimate the size of given angles and order them; draw angles reliably as parts of compound shapes

c read the time from analogue and digital 12- and 24-hour clocks to the nearest minute; use units of time (seconds, minutes, hours, days, weeks, months) and know the relationships between them; calculate time intervals from clocks, from timetables, and from calendars; use 'a.m.' and 'p.m.' and 24-hour clock notation

d recognise the need for standard units of length, area, volume, mass, and capacity, choose which ones are suitable for a given task, and use them to make sensible estimates in everyday situations; measure and weigh items; convert one unit into another [e.g. convert 3.17m to 317cm, 3.17kg to 3170g, or 2750ml to 2.75 litres]; know the rough metric equivalents of any imperial units still in daily use

e understand that area is not changed by cutting and rearranging; find areas of rectangles—initially for shapes drawn on a square grid by counting squares, then using the formula; calculate the area of shapes composed of rectangles, of right angled triangles (as half a rectangle), of a parallelogram (as '$base \times height$'), and of a general triangle; estimate the area of more general shapes by counting squares

f measure and compare capacities; understand conservation of volume for liquids; understand that volume is not changed by rearranging the constituent parts of a solid shape; find volumes of cuboids, and of simple shapes made from cuboids

g choose and use suitable measuring instruments for a task; interpret numbers and read scales with increasing accuracy—interpreting a reading that lies between two un-numbered divisions; recognise that any measurement is approximate; record measurements using decimal notation; compare readings made on different scales

Key Stage 3: Fuller version

Key Stage 3: Breadth of Study

During the key stage pupils should be taught the required *Knowledge, Skills, and Understanding* through:

a extending **place value** to arbitrary integers and decimals in working with pure number and with measures

b extending their understanding of numbers to include integers (positive and negative), fractions, and decimals

c extending **exact** arithmetic to the **standard written algorithms** for integers and decimals, and the standard procedures for calculating with fractions

d using **structural arithmetic** for efficient numerical calculation, and for algebraic **simplification** of numerical, fractional, and symbolic expressions

e representing unknowns and variables by letters; using formulae; solving linear equations; representing and interpreting points in the coordinate plane; representing and interpreting straight lines and linear equations

f engaging in tasks that develop short chains of deductive reasoning and that bring out the centrality of **proof** in number, algebra, and geometry

g drawing and measuring; using the basic 'exact in principle' **ruler and compass constructions** to complete more complicated constructions; recording results using diagrams, words, and numbers (and symbols where appropriate); *angle-chasing* (using angles on a straight line, angles in a triangle, alternate angles); developing the language and results of plane geometry; analysing more complex figures in terms of triangles

h using **exact** arithmetic to calculate precisely, or to make good **estimates** for, areas and volumes and when solving problems in 2D and 3D

i linking the language of mathematics with spoken and written English; building simple logical expressions such as "... and ...", "... or ...", "if ..., then ...", "not only ..., but also ..."; interpreting carefully crafted **realistic** problems; solving **word problems**; establishing connections between number work, measures, geometry and practical settings; distinguishing between sensible and misleading uses of mathematics

j routinely tackling familiar and unfamiliar problems, including **multi-step** and **inverse** problems [e.g. "I'm thinking of a number" and 'missing digit' problems]; recognising that mathematical operations often come in 'direct-inverse' pairs, with mastery of the harder *inverse* operation dependent on achieving robust fluency in the simpler *direct* operation
k practical work in which they use mathematical analysis [e.g. of 'centre' and spread] to draw inferences from data and consider how statistics can be used to inform decisions

Key Stage 3: Knowledge, Skills, and Understanding

Teaching should ensure that appropriate connections are made between the section on *Number and algebra* and the section on *Geometry.*

Key Stage 3: Ma1: Number and algebra

1. *Using and applying 'Number and algebra'*

Pupils should be taught to:

Problem solving

a extract and use numerical, geometrical, and logical information in analysing data and tables, and in solving simple problems expressed in words
b make **connections**; use the arithmetic of integers, decimals, and fractions when solving problems in mathematics and in other contexts
c regularly solve **multi-step** problems and **inverse** problems—identifying intermediate steps, and completing the steps efficiently and reliably; when stuck, look for an approach which works
d solve problems involving measures; solve problems involving rates and compound measures; solve problems involving **ratio and proportion**; use knowledge of **exact** arithmetic to make suitable mental **estimates**

Communicating

a use spoken and written language, notation, diagrams, terminology, and symbols correctly
b recognise when information is presented in a misleading way

c present results and solutions to problems clearly, declare unknowns explicitly, and lay out solutions and **proofs** logically **line-by-line**; explain reasoning, methods, and conclusions
d interpret tables, lists, and information presented graphically; construct and interpret frequency tables, including tables for grouped data; represent and interpret data; use precise measures of 'centre' and spread; refine ways of recording

Reasoning

a understand that some statements are **exact** or correct—and can be clearly explained or **proved**, that other statements may be false—and can be shown to be false (and that the validity of some statements may have to be left unresolved for the time being)
b routinely use **place value** and **structural arithmetic** to **simplify** calculations and expressions; recognise and use the fact that mathematical operations often come in 'direct-inverse' pairs
c use basic results and step-by-step deduction to draw conclusions—in numerical, algebraic and geometrical calculations, in geometrical constructions and deductions, and in solving equations
d develop logical thinking; search for patterns, investigate apparent patterns, and test the validity of statements— proving or disproving these statements conclusively where possible [e.g. "there are exactly twenty five prime numbers less than 100"; "the sum of two squares can end in any units digit, but can never end in '11' "; "any integer has the same remainder as its digit-sum when you divide by 9"]
e explain their methods and reasoning when solving problems involving number, data and measures

2. Numbers, the number system, structural arithmetic, simplification, and algebra

Pupils should be taught to:

Counting and numbers

a count reliably forwards and backwards across hundreds and thousands boundaries (using different step lengths)
b solve problems involving **counting** [e.g. How many pages from page 171 to 263?—inclusive and exclusive; How many chords are there joining 10 points on a circle?]
c use **place value** in representing integers to 1 000 000, and deci-

mals with up to four decimal digits; express position as a 'power of 10'; choose the power of 10 to transform a given decimal to an integer (by multiplying)

Sequences; powers and roots

a recognise multiples of 2, 4, 5, 10, 20, 25, 50, 100; factorise instantly any output from multiplication tables to 10×10; recognise (or quickly test) prime numbers to 100 and test possible primes up to 500; recognise square numbers to 20×20; find all the factors of a given integer; solve problems involving triangular numbers

b recognise powers of 2 (up to 1024), powers of 3 (up to 243), and powers of 5 (up to 625); recognise square and cube roots of familiar squares and cubes; understand and find, or estimate, the square root of any positive number x (as that *positive* number whose square is equal to x); use index notation for small positive integer powers in simple numerical and algebraic expressions; recognise simple instances of the index laws

c find specified terms of a sequence given a term-to-term or a position-to-term rule; guess the simplest position-to-term rule for the n^{th} term given the first few terms of a sequence

Integers and decimals

a read, write (in figures and words) and order whole numbers and decimals with up to six digits; understand, use, and calculate freely with (positive and negative) integers; order a set of positive and negative integers and decimals, or measurements; use correctly the symbols $=$, $\neq$, $<$, $\leq$, $>$, $\geq$, $\approx$ ('approximately equal to'), etc. and the associated language

b use correctly the terms factor, multiple, common factor, common multiple; find the *hcf* and *lcm* of two given integers; use *hcf*s and *lcm*s to solve **word problems**

c multiply, and divide, any integer or decimal by 10, 100, 1000, or 10 000; know the multiplicative complements for 10 (2×5), for 100 (2×50, 4×25, 5×20), and for 1000 [e.g. 8×125], and the corresponding decimals [e.g. $\frac{1}{2} = 0.5$, $\frac{1}{4} = 0.25$, $\frac{1}{5} = 0.2$, $\frac{1}{8} = 0.125$]; recognise as alternative representations the decimal and fraction forms of simple fractions

d express any given large number as a number less than 10 times a power of 10, and a small number as a number greater than or equal to 1 times a power of 10

e compare measurements in various contexts; round integers and decimals to the nearest 'hundred', 'ten', integer, tenth, hundredth, etc.

Fractions, percentages and ratio

a understand general fractions in terms of unit fractions; locate fractions on a number line; switch freely between mixed numbers (with fractional part < 1) and standard fractional form $\frac{p}{q}$

b find fractional parts of shapes and quantities, and recognise the fractional part represented; find the fraction describing the sum of two fractional parts of the same whole; solve simple **ratio** problems

c understand equivalent fractions; express two given fractions with a common denominator; simplify a given fraction by cancelling common factors; order a list of integers and fractions given in standard form $\frac{p}{q}$

d understand 'percentage' as a fractional operator with denominator 100; find fractions and percentages of given quantities, and express part of a given whole as a percentage; express one quantity as a fraction of another; use percentages of a fixed whole for comparing parts of that whole; use the multiplicative character of percentage as an operator [e.g. in calculations involving percentage increase and decrease, and in finding the original price given the price inclusive of VAT]; distinguish between absolute and relative increase and decrease

e reduce a **ratio** to its simplest form, and establish the connection with 'fractional parts'; divide a given quantity into two parts in a given *ratio*; solve problems involving 'internal' and 'external' *ratios*, and direct and inverse **proportion** [e.g. at constant speed, distance covered is proportional to time, but for a given distance speed is *inversely* proportional to time]

3. *Calculation*

Pupils should be taught to:

Number operations and mental methods

a reinforce and extend existing mental calculation to include decimals and fractions

b use the four number operations, including inverses, flexibly in straight calculations and in solving problems

Structural arithmetic

a use multiplication tables freely, and the associated inverse steps, to simplify fractional expressions [e.g. $\frac{6\times 15}{18\times 5} = \ldots$]; convert fractions to decimals and terminating decimals to their simplified fraction equivalents; convert a fractional part of a whole to a percentage, and vice versa

b obtain the prime-power factorisation of a given integer by successive division

c understand and use **place value**, inverse operations [e.g. cancellation], and **structural arithmetic** (including the commutative, associative and distributive laws—if without the names) to **simplify** calculations [e.g. $73+48+27 = \ldots$, or $3\times 17+7\times 17 = \ldots$, or $\frac{6\times 15}{10} = \ldots$, or $16 \times 17 - 3 \times 34 = \ldots$]

d use the idea of choosing a suitable 'common denominator' to add, subtract, multiply and divide fractions; know that multiplying by $\frac{1}{n}$ is the same as dividing by n, and that dividing by $\frac{1}{n}$ is the same as multiplying by n

e solve **word problems** involving rates and **ratios**, including the **unitary method** [e.g. Suppose n items cost £p. What does 1 item cost? How many items do you get for £1?]

f use the fact that square and square root are partial inverse operations to give both roots of simple quadratic equations [e.g. $x^2 = 9$, $x^2 = 8$, $x^2 = 10$]; use the definition of square root to identify different- looking surds [e.g. in solving $x^2 = 8$, notice that $\sqrt{8}$ is the same as $2 \times \sqrt{2}$—since both are positive and have square 8]; *simplify* expressions involving simple surds [e.g. in calculating the length of the hypotenuse of a right angled triangle with legs of lengths 3 and 4, or 2 and 2]

Algebraic simplification

a substitute given numerical values into formulae to establish the link between variables; substitute given numerical values into simplified and unsimplified linear and quadratic expressions [e.g. $(x + 1)^2$, $x(x+2)+1$, and x^2+2x+1; or $(x+y)^2$ and x^2+y^2; or x^2-y^2 and $(x-y)(x+y)$]

b multiply out brackets, collect like terms, identify and take out common factors to simplify expressions [e.g. $6(a-b)+3(2b-a) = \ldots$]; recognise that different-looking expressions may be identical [e.g. $6(a - b) +$

$3(2b-a)$ and $3a$, $(a+b)^2$ and $a^2+2ab+b^2$, a^2-b^2 and $(a-b)(a+b)$]; prove that two given expressions are identical (by algebraic transformation), or that they are not identical (by substituting values)

Written methods

a relate decimal arithmetic to integer arithmetic; use **standard written methods** in column format for addition and subtraction, short and long multiplication, short {and long} division of integers and decimals; use approximations and other strategies to check that answers are reasonable

b solve inverse problems related to *standard written methods* [e.g. 'missing digit' problems]

Inequalities

a solve simple linear inequalities in one variable and represent solutions on a number line

Measures

a calculate and work with perimeters, areas, volumes, durations, capacities; work with speed and other compound measures; use standard units of length, mass, and capacity; convert between related units

b estimate the size of any given angle; draw and measure angles reliably to the nearest degree

c calculate reliably with measures; interpret the level of accuracy by specifying the 'actual' measurement in terms of inequalities [e.g. "$s = 10.2$ cm" as indicating "10.15 cm $\leq s <$ 10.25 cm"], and using 'sig figs'; extract and use information from tables and charts; solve **word problems** involving money, time, length, and compound measures (speed, rates)

d read scales accurately—with appropriate rounding; record and order measurements using decimal notation and 'sig figs'; compare readings made on different scales

4. *Algebra: equations, formulae, identities, and functions*

Pupils should be taught to:

a use letters for unknown numbers; rearrange and solve linear equations in complete generality i.e. in which the unknown and constants appear on both sides [e.g. $\frac{1}{2}-\frac{3}{4}\cdot x = \frac{(2-4x)}{5}$, or $\frac{3}{2-4x} = \frac{7}{3x-1}$]; reduce any linear equation in two variables to standard form

($ax + by = c$, or $y = mx + c$); eliminate a variable from two simultaneous linear equations in two unknowns with general coefficients

b use letters as variables standing for 'any number'; set up and use general formulae; change the subject of a formula; draw the graph of a linear function, identifying its gradient, and interpreting its position in terms of change of variables [e.g. interpreting $y = 2x + 1$ as the function '$y = 2x$' shifted "up $+1$" or "down -1", and as the function '$y = 2x$' shifted "left $+\frac{1}{2}$" or "right $-\frac{1}{2}$"]; construct linear functions arising from real problems [e.g. where cost is proportional to amount, after a fixed initial charge; or relating temperatures in Celsius and Fahrenheit], plot and interpret their graphs; establish the link to **ratio** and **proportion**

c use letters in general expressions; use index notation for small positive integer powers; develop the algebra of expressions (collecting up like terms; multiplying out brackets; taking out common factors; etc); **simplify** given expressions

d {use algebra to find the exact solution of two simultaneous linear equations in two unknowns by eliminating a variable}

e {tabulate values and sketch the graph $y = f(x)$ for simple quadratic functions; set up and solve simple quadratic equations (by factorising the LHS of $f(x) = 0$); relate the solutions to the x-intercepts of the graph of $y = f(x)$}

5. *Solving numerical problems*

Pupils should be taught to:

a use the four number operations and powers and roots to solve arithmetical problems, **word problems**, and geometry problems involving numbers, or money, or measures of length, area, volume, mass, capacity, time, speed, and rates

b choose and use an appropriate calculation method when faced with a problem or puzzle; explain their methods and reasoning

c solve **multi-step** and **inverse** problems with confidence

d invent appropriate **word problems** which embody given calculations

e estimate answers by approximating; check that their results are reasonable by thinking about the context of the problem, and where necessary checking accuracy; explain why their answers are correct

f use algebraic formulae; set up and solve equations

6. *Processing, representing and interpreting data*

Pupils should be taught to:

a solve problems involving simple lists, tables, and charts; sort, classify, and organise information; use pictograms, pie charts, bar charts, and line graphs; discuss their methods and explain what they find

b find, and think in terms of, 'centres' (mean, median, mode) and spread (range, quartiles, interquartile range) for small datasets; identify the modal class for grouped data; interpret frequency diagrams and histograms; use cumulative frequency

c use **counting** in carefully chosen symmetrical examples [e.g. rolling one or two perfect cubical dice], where each outcome is 'equally likely', to introduce ideas of **probability** as a number between 0 and 1; explore the complement of an event, and compound events.

Key Stage 3: Ma2: Geometry

1. *Using and applying 'Geometry'*

Pupils should be taught to:

Problem solving

a work with and use triangles, rectangles, parallelograms, rhombuses, quadrilaterals, polygons, circles, cuboids, prisms, cylinders, pyramids, etc. in tackling and solving geometrical problems involving angles, length, area, and volume; know their properties and select and perform appropriate calculations

b measure and calculate accurately to construct and analyse 2D and 3D figures; use standard units of measurement arising in applications of geometry; convert reliably between related units

Communicating

a use geometrical language, notation, terminology, and symbols correctly (line *segment* $\underline{AB}$, $\triangle ABC$ as an *ordered* triple, $\angle ABC$ as the angle between lines BA and BC at B, $ABCD$ as a 'cyclically labelled' quadrilateral, etc.)

b present and interpret solutions to problems involving geometrical figures and measures c work freely in all four quadrants of the coordinate plane—using arithmetic and algebra for positive and negative numbers and linear expressions

d organise work and record results clearly—with calculations, constructions, and **proofs** laid out **line-by-line** (giving explicit justification where it is needed)

Reasoning

a distinguish between an example and a **proof**

b use basic geometrical principles [e.g. 'vertically opposite angles equal', or 'alternate angles equal', or 'angles in a triangle sum to 180º, or 'SAS congruence'] to justify each step in a calculation or deduction [e.g. for *angle-chasing*; calculation of length, area or volume, or problems involving parallel lines]

c analyse 2D and 3D configurations in terms of triangles

2. *Constructing and analysing geometrical configurations*

Pupils should be taught to:

Know and analyse

a recognise right angles, perpendicular and parallel lines, and use the associated language precisely [e.g. 'alternate angles']; know that angles at a point total 360°, that angles at a point on a straight line total 180°; estimate by eye the size of a given angle in degrees

b know that two lines are parallel precisely when alternate angles are equal (or equivalently, when corresponding angles are equal); prove, and use, immediate consequences (such as: that the angles in any triangle total 180°, that the exterior angle of any triangle is equal to the sum of the two interior opposite angles, that the angle sum in any quadrilateral is 360°, and that opposite angles in any parallelogram are equal)

c use known angles and angle properties to find unknown angles in given configurations (i.e. *angle-chasing*)

d use the perimeter of a square and of a regular hexagon inscribed in a given circle of radius r to motivate the formula for the circumference of the circle and to get a lower bound for π; solve related problems

e talk clearly about common 2D and 3D shapes (including triangles [e.g. right-angled, isosceles, and equilateral] and quadrilaterals [e.g. trapezia, parallelograms, rhombuses, rectangles and squares]; visualise 3D shapes from 2D drawings; correctly copy drawings from the board; make and draw shapes with increasing accuracy, and analyse their geometrical properties

Constructions and congruence

a use ruler and protractor to draw triangles with given data; recognise when the resulting triangle is unique (SAS, SSS, ASA, SAA; RHS) and when it is not (ASS, AAA); extract and apply the basic congruence criteria (SAS, SSS, ASA; RHS) to prove standard results (such as: base angles of an isosceles triangle are equal—and the converse; opposite sides of a parallelogram are equal, and the diagonals bisect each other; the bisector of the apex angle in an isosceles triangle is also the perpendicular bisector of the base)

b draw specified figures using 'ruler' (i.e. *straightedge*) and compasses only

c understand the basic 'exact in principle' **ruler and compass constructions** (construct the perpendicular bisector of a given line segment $\underline{AB}$; erect/drop a perpendicular from a point to a line; bisect any given angle) and use these to complete other constructions

Area and Pythagoras

a find the area of rectangles and shapes made from rectangles; find the area of a general parallelogram (as 'base × height'), the area of a right angled triangle (as half a rectangle), and the area of a general triangle (as $\frac{1}{2}$(base × height)

b find upper and lower bounds for areas of other shapes [e.g. circles] by 'counting squares', or considering inscribed and circumscribed polygons

c relate the formula for the area of a circle to the formula for the circumference; use the area of the inscribed square and regular hexagons to obtain a crude estimate for π; use the formula to solve problems

d state, prove, and use *Pythagoras' Theorem*; apply this to calculate the diagonal of a square of side 1, and the height and area of an equilateral triangle of side 2

Circles

a understand and use the terms centre, radius, chord, diameter, circumference, tangent, arc, sector, segment

b **prove** the basic properties of a circle [e.g. centre and any chord form an isosceles triangle; angle in a semicircle is a right angle; the tangent to a circle at a point P is perpendicular to the radius at P (or get an isosceles triangle with two right angles); deduce that the two tangents from any exterior point are equal]; apply these results to solve problems

Volume and 3D

a calculate volumes of cuboids and shapes made of cuboids; calculate volumes of a 'wedge' (half a cuboid), polygonal right prisms, and cylinders

b find lengths and angles in simple 3D figures by considering 2D cross-sections

Scaling and enlargement

a draw figures to scale; interpret distances, angles, and areas on maps and other scale drawings

Loci

a interpret a circle centre O passing through a given point A as the locus of (all) points X such that $OX = OA$; interpret the perpendicular bisector of a given line segment $\underline{AB}$ as the locus of (all) points X such that $AX = BX$; {interpret the two angle bisectors of two crossing lines AB, AC as a locus of points equidistant from AB and AC}

3. *Coordinates and graphs*

Pupils should be taught to:

a read and plot coordinates—eventually in all four quadrants; draw, or locate, shapes with given properties in the coordinate plane

b use *Pythagoras' Theorem* to calculate the distance between two given points (simple cases); find the coordinates of the midpoint of a line segment (simple cases)

c establish the link between straight lines in the coordinate plane and linear equations in x and y; find the gradient of a given straight line; understand that parallel lines have the same gradient; find the intersection of two given straight lines; apply these ideas to solve problems

d sketch the graphs of simple quadratic functions

e explore and use coordinates in 3D

Key Stage 4: Fuller version

Key Stage 4: Breadth of Study

During the key stage pupils should be taught the required *Knowledge, Skills, and Understanding* through:

a activities that consolidate, strengthen, and extend material from previous curriculum phases, so that it can be used efficiently and flexibly; moving on to achieve **fluency** and **automaticity** in using a wide range of arithmetical, algebraic, trigonometrical, and geometrical procedures

b using language, terminology, and logic precisely and correctly

c solving **word problems**; linking the language of mathematics with spoken and written English via carefully crafted **realistic** problems

d **learning by heart** and using flexibly that limited collection of **basic facts and techniques** in terms of which most problems at this level can be solved

e routinely tackling two-step and **multi-step exercises** and **problems**; using basic techniques in different contexts; being expected to grapple with simple, unfamiliar problems which can be effectively solved using available tools

f becoming increasingly aware of, and routinely exploiting the **connections** that link superficially different topics, and that lead to the **compression** of ideas and techniques (so reducing the number of things that have to be remembered)

g recognising that mathematical operations often come in 'direct-inverse' pairs, that the **inverse** operation is often the more demanding one, and that its mastery depends on robust fluency in the direct operation

h discussing and distinguishing between sensible and misleading uses of mathematics

i using **calculators** intelligently where needed, whilst avoiding inappropriate dependence

j extending **exact** arithmetic (without **calculators**)— originally learned for integers and decimals—to fractions, surds, and numerical and algebraic expressions involving powers and quotients; using algebraic structure to routinely **simplify** numerical, fractional, and symbolic expressions; using important structural laws to extend definitions [e.g. in interpreting x^0, or in assigning a negative value to $\cos 120°$]

k appreciating the need for **approximation** in working with measurements and real data, and handling the associated calculations reliably

l linking number, algebra, and geometry in working with coordinates, graphs, and curves

m combining congruence and **ruler and compass constructions** to establish a **formal** basis for elementary **Euclidean geometry**—later incorporating parallels and similarity

n practical and theoretical work in which they interpret standard types of tables and lists, and information presented graphically; distinguishing between numerical data representing a single idealised measure [e.g. "the height of Nelson's column"] and uncertain data sampled from a **population** or distribution [e.g. "the height of a UK adult male in 2010"]; drawing inferences from a mathematical analysis of data; considering how statistics can be used to inform decisions

Key Stage 4: Knowledge, Skills, and Understanding

Teaching should ensure that appropriate connections are made between the section on *Number and algebra* and the section on *Geometry*.

Key Stage 4: Ma1: Number and algebra

1. *Using and applying 'Number and algebra'*

Pupils should be taught to:

Problem solving

a use numerical, algebraic, geometrical, and logical information in tackling problems in *Number and algebra*, in solving **word problems**, and in analysing data

b use the structure of arithmetic and the laws of algebra to calculate efficiently when working with integers, decimals, fractions, surds, and algebraic expressions in solving problems

c regularly solve **multi-step** problems and **inverse** problems

d make use of relevant **connections** between topics

e solve problems involving measures, rates and compound measures, **ratio and proportion** f make and justify **estimates**

Communicating

a use spoken and written language, notation, diagrams, terminology, and symbols correctly
b recognise when information is presented in a misleading way
c present results and solutions to problems clearly, declare unknowns explicitly, and lay out solutions and **proofs** logically **line-by-line**
d interpret tables, lists, and charts; present information graphically
e construct and interpret frequency tables; use precise measures of 'centre' and spread

Reasoning

a investigate apparent patterns; generate, interpret, test, and prove (or disprove) simple conjectures
b use **place value**, index laws, and **structural arithmetic** to **simplify** calculations and expressions, and to justify the extension of known conventions [e.g. explaining why we set $(-1) \times (-1) = 1$, $2^0 = 1$, $\cos 120° = -\frac{1}{2}$]
c use known results and step-by-step deduction to draw conclusions

2. ***From numbers to algebra (including calculation)***

Pupils should be taught to:

Numbers and arithmetic

a use **place value** in representing and calculating with arbitrary integers and decimals; work effectively with problems involving very large numbers;
b multiply, and divide, any integer or decimal by any power of 10; know the multiplicative complements for 10 (2×5), for 100, and for 1000, and the corresponding decimals [e.g. $\frac{1}{2} = 0.5$, $\frac{1}{5} = 0.2$, $\frac{1}{4} = 0.25$, $\frac{1}{8} = 0.125$]; recognise as alternative representations the decimal and fraction forms of simple fractions
c understand and use **divisibility tests** for 2, 3, 4, 5, 8, 9, 10, and 11, and deduce those for 6 and 12
d understand informally why $(-1) \times (-1) = 1$, later understand how "$0 = (-1) \times 0$" and the distributive law imply "$0 = (-1) \times 0 = (-1) \times [1 + (-1)] = (-1) \times 1 + (-1) \times (-1)$", from which it follows that "$(-1) \times (-1) = 1$"; work freely and accurately with

expressions involving positive and negative integers, decimals, fractions, and surds—**simplifying** routinely before evaluating

e choose the power of 10 that will transform a given terminating decimal into an integer by multiplying; express any given large number as a smaller number times a positive power of 10, and a small number as a larger number times a negative power of 10 (preparing for standard form)

f solve problems involving **counting** [e.g. How many pages from page 171 to 263?—inclusive and exclusive]; understand and use the **product rule** for *counting* [e.g. if 7 wives each have 7 sacks, and each sack contains 7 cats, then the total number of cats is $7 \times 7 \times 7$]

g consolidate and extend short and long division, prior to exploring links between fractions and decimals

Measures

a compare measurements in various contexts; round integers and decimals appropriately; use $\approx$ ('approximately equal to') where appropriate

b calculate and work with perimeters, areas, volumes, durations, capacities; use standard units of length, area, volume, mass, and capacity, and simple compound measures (speed, density, and other 'rates'); read scales with appropriate rounding; record and order measurements using decimal notation; change freely between related units – in numerical and algebraic contexts

c draw up tables and charts; extract and use information from tables and charts; solve **word problems** involving money, time, length, and compound measures

d calculate reliably with measures; solve problems involving measures [e.g. If lamp posts are 50m apart, how far is it from the first to the tenth lamp post?]

Bounds and estimation

a understand and interpret as appropriate the limits of accuracy implied by a given measurement in decimal form [e.g. '2.13cm' represents a measurement < 2.135cm and ≥ 2.125cm], and relate this to 'sig figs'; interpret the result of an arithmetical calculation, rounding up or down as needed [e.g. if each pen costs 60p, then £3.50 will buy 5 pens; but if each bus can take 60 children, then to transport 350 children we require 6 buses]

b establish bounds on the accuracy of an **estimate**; appreciate exactly how bounds on the accuracy of the inputs to an approximate calculation affect the possible error in the output from the calculation [e.g. when adding lengths to calculate the perimeter of a rectangle with given approximate dimensions, or when multiplying to find the area]; calculate effectively using estimates (without a calculator)

Integer factorisation, fractions, and surds

a use correctly the terms factor, multiple, common factor, common multiple; find and use the *hcf* and *lcm* of two (and hence three or more) given integers; solve related **word problems**

b recognise (or test quickly) prime numbers up to 120; understand and use the 'square root test' ("if an integer factorises $N = a \cdot b$ with $a, b > 1$, then the smaller of the two factors is $\leq \sqrt{N}$") to identify possible primes up to 1000 and beyond

c obtain the prime power factorisation of a given integer (using successive division); hence list all factors of a given integer; solve related problems

d move freely between mixed fractions (with fractional part < 1 [e.g. $3\frac{3}{4}$]) and fractions in standard fractional form $\frac{p}{q}$ [e.g. $\frac{15}{4}$]; simplify (i.e. reduce to lowest terms) a given fraction; rewrite two given fractions with a common denominator; compare and order pairs of fractions (initially by reducing both to a common denominator, later interpreting this process in terms of 'cross-multiplying')

e use factorisation to simplify surd expressions [e.g. $\sqrt{12} = \sqrt{2^2 \times 3} = 2\sqrt{3}$, since all are positive and have square equal to 12; $\sqrt{12} \times \sqrt{6} = 6\sqrt{2}$, $2\sqrt{3} \times \sqrt{6} = 6\sqrt{2}$; $\frac{\sqrt{108}}{\sqrt{150}} = \frac{3\sqrt{2}}{5}$]

Fractions

a understand the unit fraction $\frac{1}{q}$ as "that part, of which q identical copies make 1"; understand a general fraction $\frac{p}{q}$ as a multiple $p \times \frac{1}{q}$ of a unit fraction; routinely **simplify** numerical fractions in standard form $\frac{p}{q}$ or in factorised form [e.g. cancelling

$$\frac{2 \times 3 \times 5}{9 \times 10} = \frac{1}{3}$$

without first evaluating numerator and denominator]; move freely from a given fraction to a suitable equivalent fraction

b add and subtract fractions; divide fractions (initially by reducing the numerator and denominator to multiples of a common unit fraction, later interpreting this as 'invert and multiply'); multiply unit fractions—and hence general fractions (including multiplication by integers [e.g. $17 \times \frac{2}{3} = \ldots$])

c *simplify*, and hence evaluate, compound expressions involving fractions—including combinations of addition/subtraction and multiplication/division together with fractional expressions in which the numerator and/or denominator is/are the sum or difference of two fractions (as is required when finding the equation of the straight line through the two points $(\frac{2}{3}, -\frac{3}{4})$ and $(\frac{1}{2}, \frac{1}{3})$)

Fractions and decimals

a move freely between terminating decimals and decimal fractions (i.e. fractions with denominator a power of 10)

b know the equivalence of the **exact** (unevaluated) fraction notation $\frac{p}{q}$ and the result of evaluating $p \div q$; hence find the decimal of any given fraction; recognise that the decimal form of $\frac{p}{q}$ must terminate or recur (since there are only $q-1$ possible remainders, so some remainder must repeat) and that the recurring block has length $< q$

c change any given terminating decimal into a decimal fraction, and hence into a fraction in its lowest terms; given any recurring decimal with a recurring block of length n, multiply by 10^n and subtract to transform the given decimal into a fraction with denominator $10^n - 1$

Surds

a recognise $\sqrt{k}$ (for $k > 0$) as denoting the **exact** positive real number whose square is equal to k; given a value of k, find approximate values for $\sqrt{k}$; use the 'algebraic arithmetic' of unevaluated surds [e.g. $\sqrt{ab} = \sqrt{a} \times \sqrt{b}$, since both are positive and have square equal to ab], including rationalising denominators [e.g. $\frac{1}{\sqrt{2}+1} = \sqrt{2} - 1$]; routinely give lengths arising from applications of *Pythagoras' Theorem* and solutions to quadratic equations in *exact* (mixed surd) form; use the standard notation for, and calculate with, cube roots

b use surds and π to calculate *exactly* in geometric contexts [e.g. the height and area of an equilateral triangle of side 2; the lengths of the diagonals of a square, a regular pentagon, a regular hexagon; the area of a circle with circumference 10; the circumference of a circle with area 10]

Powers, roots, and the index laws

a factorise instantly any output from multiplication tables to 10×10; recognise square numbers to 25×25; recognise cubes to 6^3

b recognise powers of 2, 3, 4, 5; recognise square and cube roots of familiar squares and cubes; extend squares and cubes to fractions and decimals; extend square and cube roots to suitable fractions and decimals; find, or estimate, the square root or cube root of any positive number

c know, understand, and use the index laws for positive integer powers; understand that the meaning of x^a for positive fractional values of a (such as $a = \frac{1}{2}$, or $a = \frac{1}{3}$) is chosen to preserve and extend the index laws [e.g. so that $x^{\frac{1}{2}} \cdot x^{\frac{1}{2}} = x^1 = x$, etc.]; similarly define x^a for $a = 0$, and for negative values of a [e.g. $a = -1$]; **simplify** numerical expressions given in index form [e.g. $50^{\frac{1}{2}} = \ldots$, $8^{-\frac{2}{3}} = \ldots$, $(\frac{27}{216})^{-\frac{2}{3}} = \ldots$]; use index notation to present numerical and algebraic expressions in simplified power form; calculate freely with (unevaluated) numerical and algebraic expressions involving powers

d write any given number in standard form; translate a given standard form into the (approximate) number it represents; understand and use 'sig figs'; calculate with numbers given in standard form as though they are 'exact' [e.g. $(2.4 \times 10^7) \times (5 \times 10^3) = 1.2 \times 10^{11}$; $(2.4 \times 10^6) + (5 \times 10^3) = 2.405 \times 10^6 \approx 2.4 \times 10^6$; $(2.4\times10^6)-(5\times10^5) = 1.9\times10^6$; $(2.4\times10^6)\div(5\times10^3) = 4.8\times10^2$]; solve problems involving standard form

e {multiply powers with a fixed base "a" by adding exponents; introduce and use logarithms *base 2* and *base 10*}

Fractions, decimals, and percentages

a find fractional parts of shapes and quantities, and recognise the fractional part represented; express one quantity as a fraction of another

b understand percentage as a fractional operator with denominator 100; know and use the percentage equivalents of familiar fractional parts; work freely with percentages

c understand and use the multiplicative character of percentage increase and decrease [e.g. know that increasing a quantity A by 25% gives $\frac{5}{4}A$, or $(1.25)A$; know that decreasing A by 25% gives $\frac{3}{4}A$, or $(0.75)A$; know that the original cost £C of an item which costs £100 after adding 20% VAT satisfies $1.2\,C = 100$]; solve problems involving percentage change (including compound interest)

Sequences

a generate and work with standard integer sequences (including the sequences of odd and even numbers; numbers of the form $3n$, or of the form $3n+1$, or of the form $3n-1$; squares; cubes; powers of 2; powers of 10; triangular numbers; Fibonacci numbers)

b find specified terms of a sequence given the first term (or terms) and a term-to-term rule [e.g. $x_1 = 3$ and $x_{n+1} = 2x_n$; or $x_1 = x_2 = 1$ and $x_{n+1} = x_n + x_{n-1}$; or $x_1 = 2$ and $x_{n+1} = 3x_n - 2^{n-1}$], or a position-to-term rule [e.g. $x_n = 4n-3$, or $x_n = n^2-n+41$, or $x_n = 2^n-1$, or $x_n = 3^n+2^n$]; guess the simplest position-to-term rule for the n^{th} term given the first few terms of a sequence

c use given initial terms and a term-to-term rule for a sequence to find and prove a **closed formula** for the position-to-term rule specifying the n^{th} term in terms of n

d find and justify the term-to-term rule and the position-to-term rule for sequences **defined intrinsically** [e.g. from a sequence of patterns, or from a geometrical or counting problem with parameter n]

e interpret a sequence as a function defined only for positive integer inputs; understand how the sequence of powers (x^n) behaves "as n increases" for specific values of x with $x > 1$ {or $|x| > 1$}, and with $0 < x < 1$ {or $|x| < 1$ }; link to compound interest and population growth, to doubling times, and radioactive half-life

Ratio and proportion

a divide a given quantity into two parts in a given part-to-part, or part-to-whole **ratio**; express a given division of a quantity into two parts as a *ratio*; establish the connection with 'fractional parts'; reduce a *ratio* $a : b$ to its simplest form $c : d$ (where a, b, c, d are integers and $\text{hcf}(c, d) = 1$), and to the 'unit forms' $1 : e$ and $f : 1$; solve problems involving *ratios*

b calculate the result of a change of units; draw and use scale diagrams and maps; understand the effect of scaling and enlargement on different quantities (including angles, lengths, areas, and volumes);

c solve problems in which different quantities are **proportional**; solve proportion problems (given the values of any three of the four quantities in a *proportion*, determine the fourth [e.g. If 20 litres of petrol cost £27.60, how many litres would one get for £40? And what would I pay *for* 53 litres?"]); use the **unitary method**, and then the general method, to solve *proportion* problems

d understand and use "X is inversely proportional to Y" as meaning "X is proportional to $\frac{1}{Y}$" '

Algebraic expressions

a substitute numerical values into formulae and expressions

b multiply out brackets, collect like terms, identify and take out common factors to **simplify** linear, quadratic, and higher order expressions involving one, two, or more letters, e.g.

$$(4+x)+(2-x)=\dots,$$

$$(4+x)-(2-x)=\dots,$$

$$4x-5y+3z-(2z-6y+3x)=\dots,$$

$$8a^2bc-6c^2ba+4b^2ac-2abc(2b+4a-3c)=\dots;$$

simplify additive, multiplicative, and rational expressions (possibly involving powers and roots) by combining additive simplification, the distributive law, and cancellation—giving answers where possible in factorised form, e.g.

$$3(4x+1)+2(4x+1)=\dots,$$

$$2y(4x+1)+2(4x+1)=\dots,$$

$$y(2x+1)-z(4x+2)=\dots;$$

$$\frac{4a^2b^3}{6ab}=\dots,$$

$$\frac{y(2x+1)-z(4x+2)}{y-2z}=\dots,$$

$$\frac{4x-5y+3z-(2z-6y+3x)}{x+y+z}=\dots,$$

$$\frac{(x^2-y^2)}{(x^2+xy-2y^2)}=\dots;$$

work with algebraic fractions having linear and quadratic denominators c use the rules of algebra to rearrange formulae; solve problems using standard formulae [e.g. for areas and volumes, for average speed, for density]; find x given particular values of y in simple equations [e.g. $y = \frac{k}{x}$, or $y = \frac{k}{x^2}$]

d declare unknowns, set up linear equations arising in different areas of mathematics, and solve them; solve the general linear equation in one unknown (that is, any linear equation in which the unknown appears on one or both sides, with arbitrary coefficients and constants) presenting the solution process **line-by-line** and justifying each step

e set up a linear equation relating two unknowns x, y [e.g. $ax+by = c$, or $ax = by+c$]; interpret a linear equation in two unknowns in the coordinate plane as representing all points (x, y) on a straight line; draw the graph of a linear function, identifying its gradient, and interpreting its position; know that parallel lines have the same gradient; prove and use the fact that two lines with gradients m and m' are perpendicular precisely when $m \cdot m' = -1$; find the gradient from an equation given in any form; transform any given equation into the form $y = mx + c$ (or $x = a$); construct linear functions arising from real problems, sketch and interpret their graphs; establish the link to **ratio and proportion**

f solve any pair of simultaneous linear equations in two unknowns by eliminating a variable; interpret the analytic solution of simultaneous linear equations as 'finding the point of intersection' (if any) of the two lines

g factorise given quadratic expressions in one variable; use the fact that 'a product $p \cdot q = 0$ precisely when one of the factors p, $q = 0$' to solve quadratic equations in the form $f(x) = 0$ by factorising the LHS; interpret the solutions as those values of x where the graph $y = f(x)$ of the quadratic function crosses the x-axis; solve fractional equations that reduce to quadratics [e.g.

$$\frac{6}{x+4} = x + 3, \qquad \frac{1}{x-1} + \frac{2}{x+2} = 3]$$

h factorise the difference of two squares; use this to evaluate numerical expressions efficiently [e.g. 19^2 , or 21^2]; conclude that, if $a^2 = b^2$, then $0 = a^2 - b^2 = (a - b)(a + b)$, so $a = \pm b$, whence, for a given value of k, the equation $x^2 = k$ has two solutions $\pm\sqrt{k}$ if $k > 0$, one solution if $k = 0$, and no real solutions if $k < 0$; interpret this as a statement about the graph of $y = x^2 - k$

i know and use the expansion of $(x + a)^2$ [e.g. to check $11^2 = 121$, and to compute 101^2, 1.1^2, 1.01^2]; {extend to $(x+a)^3$, and compute 11^3, 1.1^3, 101^3, etc.;} use the expansion of $(x+a)^2$ to 'complete the square' for any given quadratic; hence obtain the formula for the solutions of the general quadratic $y = ax^2 + bx + c$; use this formula to solve given quadratics arising in different contexts (including geometry); find the maximum or minimum point and deduce the symmetry of the graph of a quadratic function about the vertical line through this point

j solve two simultaneous equations where one is linear and the other quadratic—including where the quadratic is the equation of a circle centred at the origin; interpret the result geometrically; {find the points where two circles intersect}

k understand the difference between an **equation** (true for certain restricted values of the variables) and an **identity** (true for all permissible values of the variables); decide whether two given expressions are identical or not, and **prove** they are (by transforming one into the other), or show that they are not (by substituting suitable values of the variables)

l solve linear inequalities in one and two variables; interpret the solution graphically

3. *Coordinates, graphs, and functions*

Pupils should be taught to:

a read and plot coordinates in all four quadrants; move freely between straight lines in the coordinate plane and linear equations in x and y; derive the equation of a line through any two given points, and the equation of a line through a given point with a given gradient; solve related problems

b find the coordinates of the midpoint of a line segment; use *Pythagoras' Theorem* to calculate the distance between two given points in 2D {or 3D}; show that the line segment $\underline{MN}$ joining the midpoints M, N of $\underline{AB}$ and $\underline{AC}$ is parallel to BC and half the length of $\underline{BC}$, with $\triangle AMN$ a half-sized version of $\triangle ABC$

c construct linear functions and draw the corresponding graphs; discuss and interpret straight line graphs arising in 'real situations' [e.g. distance-time graphs for a particle moving with constant speed; the depth of water in a cuboid with one dimension vertical, when filled from a tap with constant rate of flow; the velocity-time graph for a particle moving with constant acceleration]; find the gradient and intercept of a given straight line graph

d know and use the general form $y = mx+c$ (or $x = a$) for a straight line; understand and use gradient and intercept; recognise the family of parallel lines with fixed gradient "m"; recognise the family of lines concurrent at $(0, c)$ having fixed intercept "c"; solve problems requiring one to find the point of intersection of two given straight lines [e.g. in proving that the medians of any triangle ABC are concurrent]

e prove [e.g. by using similar triangles] that two lines with gradients m and m' are perpendicular precisely when $m \cdot m' = -1$; use this to find the equation of the tangent to a circle at a given point

f for particular values of m and c interpret the standard form $y = mx + c$ as '$(y - c) = mx$' (that is, as '$Y = mx$' moved "up $+c$"), and as '$y = m(x + \frac{c}{m})$' (that is, as '$y = mX$' moved "right $-\frac{c}{m}$")

g sketch the graphs of any given quadratic function by completing the square; use knowledge of the graph of $y = x^2$ to sketch for particular values of d, c the graphs of $y = (x+d)^2$ and $y = x^2+c$; hence, given any quadratic function $y = \pm x^2 + bx + c$ (and later given any quadratic function $y = ax^2 + bx + c$), complete the square and sketch the graph

h sketch other graphs—including simple cubic functions [e.g. $y = x^3$, $y = x^3 - x$], the reciprocal function $y = \frac{1}{x}$ (with forbidden value $x = 0$), the exponential function $y = k^x$ for easy (positive) values of k [e.g. $k = 2$ and $k = \frac{1}{2}$]; the circular functions $y = \sin x$, $y = \cos x$ {and $y = \tan x$}

i use coordinates to solve simple problems in 3D

4. *Processing, representing and interpreting data*

Pupils should be taught to:

a engage in practical and theoretical work in which they interpret tables, lists, and information presented graphically; construct and interpret frequency tables; present results using piecharts, diagrams for continuous data, scatter diagrams, frequency diagrams, histograms; solve problems involving lists, tables, charts, and graphs; sort, classify, and organise information; use precise measures of 'centre' and spread; discuss the methods used and explain what they find

b distinguish between numerical data representing a single idealised measure (such as "the height of Nelson's column") and informal 'random variables' [e.g. "the height of a UK adult male in 2010"] sampled from a **population** (or distribution); understand that random variables are quite different from single-valued measures (such as "the diameter of a 2p coin")

c discuss and make sense of simple examples of **populations**—both mathematical (such as, the results of tossing a coin, or the outcomes when rolling a pair of dice) and 'real' (such as the heights of UK adult males in 2010); select and interpret random samples from such populations; consider what inferences can be drawn from sampled data, and identify the kind of question that can be addressed by statistical methods; make simple analyses of data and grouped data; resolve difficulties presented by missing data, or "outliers"

d calculate and work efficiently with the 'average', or mean, of a set of numbers or measures; understand when other measures (mode, median) are more appropriate ways to summarise the 'centre'; find the median and mode for a given set of numbers; work flexibly to find the mean (and spread) of small datasets [e.g. treating 77, 84, 94 as $80-3, 80+4, 80+14$ with mean $80+5$; and 184, 277, 394 as $100+84, 200+77, 300+94$ with mean $200+85$]; identify the modal class for grouped data; refine measures of spread and 'central tendency'; use cumulative frequency

e introduce ideas of **probability** via simple examples of discrete sample spaces S in which each single outcome is self-evidently equally likely [e.g. tossing a single 'fair' coin $S = \{H, T\}$, with

$$\text{prob}(H) = \text{prob}(T) = \frac{1}{2};$$

or rolling a 'fair' die with

$$S = \{1, 2, 3, 4, 5, 6\},$$

each occurring with probability $\frac{1}{6}$; or randomly picking a card from a standard pack (without jokers) where S consists of the 52 cards, each occurring with probability $\frac{1}{52}$]; list all single outcomes, or 'events', in simple examples [such as a sequence of three tosses of a fair coin; or rolling two distinguishable fair dice]; use the general notion of an *event* (as being a subset of S) and recognise that the event *occurs* when the observed outcome is an element of the relevant subset [e.g. the event of achieving an 'odd' outcome when rolling a single die corresponds to the subset $\{1, 3, 5\}$ of S; the event of achieving a 'square total' when rolling two dice corresponds to the subset

$\{(1,3),(2,2),(3,1),(3,6),(4,5),(5,4),(6,3)\}$ of S; when tossing a coin four times, the event that 'no two successive tosses are the same' corresponds to the subset {HTHT, THTH}]; specify the sample space and list the possible outcomes for events described in words

f if A and B are *disjoint* events, understand why

$$\text{prob}(A \cup B) = \text{prob}(A) + \text{prob}(B);$$

use simple **counting** to calculate probabilities in discrete sample spaces [e.g. the probability of achieving a 'square total' when rolling two dice is $\frac{7}{36}$;
the probability of any particular sequence occurring when a coin is tossed four times is

$$\left(\frac{1}{2}\right)^4 = \frac{1}{16},$$

and the probability that 'no two successive tosses are the same' in such a sequence of four tosses is $\frac{2}{16}$]; understand and use the inclusion/exclusion formula

$$\text{prob}(A \cup B) = \text{prob}(A) + \text{prob}(B) - \text{prob}(A \cap B)$$

for events which are not necessarily disjoint

Key Stage 4: Ma2: Geometry

1. *Using and applying 'Geometry'*

Pupils should be taught to:

Problem solving

a know and understand basic **ruler and compass constructions**; use these to devise other constructions

b use standard units in geometry; use standard formulae, *Pythagoras' theorem*, trigonometry, congruence, parallels, and similarity to solve geometrical problems involving coordinates, angles, length, area, volume, polygons, circles, and familiar 3D figures

c measure and calculate accurately to construct and analyse 2D and 3D figures in terms of triangles; use known results to construct simple **proofs**

Communicating

a use geometrical language, notation, terminology, and symbols correctly

b work in all four quadrants of the coordinate plane; interpret a given equation as the graph of a function or a circle

c lay out calculations, constructions, and **proofs** in **line-by-line** format

Reasoning

a use the congruence and similarity criteria for triangles, the condition for parallels, and results derived from them to justify each step in a calculation, construction, or deduction

b analyse 2D and 3D configurations [e.g. by singling out, and using known properties of triangles]

2. *From naïve construction to Euclidean geometry*

Pupils should be taught to:

Ruler and compass constructions revisited and organised

a know that two given points A, B determine an infinite line AB, a line segment $\underline{AB}$, and a circle with centre A and radius $\underline{AB}$; recognise that constructions with ruler and compasses embody these ideal constructions; given A, B, construct an equilateral triangle $\triangle ABC$

b know and use the conventional notation for the angles and sides in $\triangle ABC$, where $\angle B = \angle ABC$, $\angle C = \angle BCA$, $\angle A = \angle CAB$, and b denotes the length of the side $\underline{CA}$ opposite $\angle B$, c denotes the length of the side $\underline{AB}$ opposite $\angle C$, a denotes the length of the side $\underline{BC}$ opposite $\angle A$

c recognise [e.g. by experience in constructions with ruler and protractor] that certain sets of data about a triangle determine its shape uniquely, while others do not; accept and use the SAS, SSS, ASA (and later RHS) congruence criteria as the initial assumptions in organising basic geometrical experience (that is, prior to knowing 'the sum of angles in a triangle'); use these criteria to **prove** the basic properties of any isosceles triangle ABC with $AB = AC$ (that the base angles are equal, $\angle B = \angle C$; that the line joining the apex A to the midpoint M of the base $\underline{BC}$ is perpendicular to the base and bisects $\angle A$); use this to implement and to justify the basic **ruler and compass constructions** ('bisect any given angle $\angle CAB$', 'bisect any given line segment $\underline{BC}$', 'construct the perpendicular to a given line BC from a given point A', where the point A may lie on the line BC or not); devise and implement related constructions; *prove* that if in $\triangle ABC$ we have $\angle B = \angle C$, then $AB = AC$

d *prove* that the perpendicular bisector of a given line segment $\underline{BC}$ is precisely the locus of all points X that are 'equidistant from' both B and C; conclude that the perpendicular bisectors of the three sides of any triangle ABC are concurrent in a point equidistant from the three vertices—namely the circumcentre of triangle ABC; solve related problems and devise and implement related constructions and *proofs*

e recognise the 'perpendicular distance' from a point X to a line as the (shortest) distance from X to the line; *prove* that any point X on the bisector of a given angle $\angle BAC$ is equidistant from the lines AB and AC; {conclude that the angle bisectors of the three angles of any triangle ABC are concurrent in a point equidistant from the three sides of the triangle—namely the incentre;} solve related problems and devise and implement related constructions and *proofs*

f {given $\triangle ABC$, prove that the three altitudes are concurrent (draw lines through A parallel to BC, though B parallel to CA, and through C parallel to AB to form $\triangle XYZ$, whose perpendicular bisectors are the altitudes of $\triangle ABC$)}

Parallel lines and angles in a triangle

a know that angles at a point total 360°, and that angles at a point on a straight line total 180°; deduce and use the fact that 'vertically opposite angles are equal'

b recognise the second basic assumption that 'two lines are parallel precisely when alternate angles (or equivalently, when corresponding angles) created by any transversal are equal'; conclude that opposite angles of a parallelogram are equal, that opposite sides of a parallelogram are equal, and that the two diagonals of a parallelogram bisect each other; **prove** that the diagonals of a rhombus are perpendicular and bisect the angles at each vertex; know and use {and prove} the *Midpoint Theorem* {in $\triangle ABC$, if M is the midpoint of $\underline{AB}$ and N is the midpoint of $\underline{AC}$, prove [e.g. by drawing the line through M parallel to BC and proving that it meets AC at N] that $MN \| BC$ and $MN = \frac{1}{2} \cdot BC$}; solve related problems c *prove* that the angles in any triangle add to 180º (given any triangle ABC, if we draw the line XAY through A parallel to BC with X and C on opposite sides of AB, then $\angle XAB = \angle CBA$ and $\angle YAC = \angle BCA$, so the three angles of triangle ABC add to $\angle XAY = 180°$); deduce that the exterior angle at any vertex is equal to the sum of the two interior opposite angles; solve a range of related problems; deduce also that that the angles in any quadrilateral add to 360°; conclude that a parallelogram with one right angle has four right angles—so is a rectangle; solve a range of related problems; {calculate the angle-sum in an n-gon, and the angle size in a regular n-gon}

d combine known results about vertically opposite angles, angles on a straight line and at a point, alternate angles, and angles in a triangle to find unknown angles in various configurations, and to show that certain pairs of lines are parallel [e.g. the diagonal AC and the side ED in a regular pentagon $ABCDE$]

e know and use the fact that in a circle with centre O the tangent at B is perpendicular to the radius OB; conclude that tangents from an external point are equal; use the fact that in a circle with centre O and chord $\underline{AB}$, $\triangle OAB$ is isosceles to *prove* that, if BC is a diameter, the angle $\angle BAC$ subtended at the point A on the semicircle is a right angle (since $\triangle OBA$ and $\triangle OCA$ are isosceles); extend this to *prove* that, if $\underline{BC}$ is a chord in any circle, the angle subtended by $\underline{BC}$ at a point A on the major arc is half the angle subtended by $\underline{BC}$ at the centre O; conclude that angles subtended by any chord in the same segment are equal, and that opposite angles of a cyclic quadrilateral add to 180º; *prove* that the angle between a tangent and a chord is equal to the angle subtended by the chord in the opposite segment; solve related problems

f *prove* that the area of a parallelogram is equal to that of a rectangle on the same base and between the same parallels; understand that a diagonal splits the parallelogram into two congruent triangles, so the area of each triangle is exactly half that of the parallelogram, and hence exactly half that of the rectangle on the same base and between the same parallels

g prove *Pythagoras' Theorem* (given triangle ABC with a right angle at A, construct the square $ABRS$ on $\underline{AB}$ and the square $BCPQ$ on $\underline{BC}$; drop the perpendicular from A to BC, meeting BC at X and PQ at Y; then $\frac{1}{2} \cdot \text{area}(ABRS) = \text{area}(RBA) =$

area(RBC), and $\frac{1}{2} \cdot$ area($XYQB$) = area(XBQ) = area(ABQ) and triangles RBC and ABQ are SAS-congruent, so area($ABRS$) = area($XYQB$); similarly area($ACVU$) = area($XYPC$), where $ACVU$ is the square on AC); solve related geometric problems involving surds [e.g. find the height and area of an equilateral triangle of side 2, or of side s; find the height and area of other isosceles triangles; find the height of a square based pyramid; find the length of the diagonals and the area of a regular hexagon of side 2], or algebraic equations [e.g. find all right angled triangles whose sides are in arithmetic progression])

Similarity

a recognise that a triangle is an *ordered* triple ABC; draw triangles with given angles and measure the sides to establish the AAA-similarity criterion for general triangles "if triangles ABC and $A'B'C'$ have $\angle A = \angle A'$, $\angle B = \angle B'$, and $\angle C = \angle C'$, then corresponding sides are in the same ratio: $BC : B'C' = CA : C'A' = AB : A'B' = k$ (say)" (that is, the first triangle is a 'k-times enlargement' of the second); denote this 'similarity relation' by $\triangle ABC \sim \triangle A'B'C'$ (with the vertices listed in the correct order); tackle relevant exercises and use the AAA-similarity criterion to solve related problems

b extend the *Midpoint Theorem* to divide a given segment into any number of equal parts; understand, **prove** and use the *Intercept Theorem* (if parallel straight lines make equal intercepts on a given transversal, then they make equal intercepts on any other transversal)

c solve problems in which similarity plays a key role [e.g. to show that the diagonal of a regular pentagon of side 2 has length $1+\sqrt{5}$]

d *prove* and use a selection of basic results using similarity [e.g. if XP is a tangent from an outside point X to the point P on a circle and XAB is a line meeting the circle at A and B, then $XAP \sim XPB$, so if XCD is another line from P meeting the circle at C and D, then $XA \cdot XB = XP^2 = XC \cdot XD$ (similarly if X is a point inside the circle and AXB, CXD are chords intersecting at X, then $XAC \sim XDB$, so $XA \cdot XB = XC \cdot XD$); solve related problems

3. Geometric calculation

Trigonometry

a work first with particular angles (say $\theta = \dots$), then with a general acute angle θ, to establish the AAA-similarity criterion for right angled triangles: "if two right angled triangles ABC and $A'B'C'$ have right angles at A and A', then if $\angle B = \angle B' = \theta$, (and hence $\angle C = \angle C' = 90° - \theta$), corresponding sides are in the same ratio: $BC : B'C' = CA : C'A' = AB : A'B'$"; conclude that the standard trig ratios for acute angles θ—namely $\sin\theta$, $\cos\theta$, $\tan\theta$—depend only on the angle θ; understand why $\sin\theta$, $\cos\theta$ take values between 0 and 1, and why $\tan\theta$ can take any positive value

b use half a square and half an equilateral triangle to find the **exact** values of $\sin\theta$, $\cos\theta$, $\tan\theta$ for $\theta = 0°$, $30°$, $45°$, $60°$; find good estimates for other values of θ, and plot the graphs of $y = \sin\theta$, $y = \cos\theta$ {and $y = \tan\theta$} for $0° \leq \theta < 90°$; notice that $\cos\theta = \sin(90° - \theta)$, and use this to tabulate values and to relate the graphs of $y = \sin\theta$ and $y = \cos\theta$; solve related problems; use a calculator to find $\arcsin x$, $\arccos x$, and $\arctan x$ for given values of x

c drop a perpendicular in a given triangle ABC, and use the altitude as a stepping stone to calculate missing lengths and angles [e.g. given $\angle A$, $\angle B$, a, b, we may drop the perpendicular CX from C to AB, create two right angled triangles, and hence calculate $c = AX + XB = b \cdot \cos A + a \cdot \cos B$; or if we are given only $\angle A$, $\angle B$, and b, we can calculate $CX = b \cdot \sin A$, so $c = AX + XB = b \cdot \cos A + \frac{b \cdot \sin A}{\tan B}$]; solve related problems

d given a triangle ABC with $\angle C < 90°$, derive and use the formula 'area$(ABC) = \frac{1}{2} \cdot ab \cdot \sin C$'; deduce the *Sine Rule* for acute angled triangles (since area$(\triangle ABC) = \frac{1}{2} \cdot ab \cdot \sin C = \frac{1}{2} \cdot bc \cdot \sin A$, so $\frac{a}{\sin A} = \frac{c}{\sin C}$ $(= \frac{b}{\sin B})$); use the *Sine Rule* to "solve triangles" directly (without the need to prop a suitable perpendicular); note that, when $\angle C = 90°$, area$(\triangle ABC) = \frac{1}{2} \cdot ab$, so the formula area$(ABC) = \frac{1}{2} \cdot ab \cdot \sin C$ suggests that "$\sin 90° = 1$"

e show that on the unit circle with centre at the origin O, the point P in the first quadrant for which the radius OP makes an angle θ with the positive x-axis has coordinates $(\cos\theta, \sin\theta)$; relate (i) the way the x- and y-coordinates vary as a point moves on the circle to (ii) the graphs of $y = \sin\theta$, $y = \cos\theta$ for $0° < \theta < 90°$, and to the value $\sin 90° = 1$; infer the value $\cos 90° = 0$; apply

Pythagoras' Theorem to derive the identity $\sin^2\theta + \cos^2\theta = 1$; use this identity to find values of $\cos\theta$ given the value of $\sin\theta$ (and *vice versa*) {and the value of $\tan\theta$ given the value of $\cos\theta$}

f check by measuring that $c^2 < a^2 + b^2$ when $\angle C < 90°$; prove the *Cosine Rule*: $c^2 = a^2+b^2-2ab\cdot\cos C$ [e.g. drop the perpendicular CX, of length h, from C to AB; then $a^2 = h^2 + XB^2$ and $b^2 = h^2+XA^2$, so $a^2-b^2 = XB^2-XA^2 = (XB+XA)\cdot(XB-XA) = c(c-2\cdot XA) = c^2 - 2c\cdot b\cdot\cos A)$]; use the *Cosine Rule* to find unknown lengths and angles in triangles and other 2D (and 3D) figures

g observe that if the formula $area(\triangle ABC) = \frac{1}{2}\cdot ab\cdot\sin C$ (or equivalently, the *Sine Rule*) is to work when $\angle C = \theta > 90°$, then we want "$\sin\theta = \sin(180° - \theta)$"; observe that for the *Cosine Rule* to work when $\angle C = 90°$, we need $\cos 90° = 0$, and for the *Cosine Rule* to work when $\angle C = \theta > 90°$, we need "$\cos\theta = -\cos(180° - \theta)$"; understand that that this is consistent with the requirement that, for the unit circle with centre at the origin, the radius OP in the second quadrant making angle θ with the positive x-axis has coordinates $(\cos\theta, \sin\theta)$; find values of $\sin\theta$ and $\cos\theta$ for $\theta > 90°$ by using given or known values for $\theta < 90°$; use the *Sine Rule* and *Cosine Rule* to find unknown lengths and angles in obtuse angled triangles; extend the graphs of $y = \sin\theta$, $y = \cos\theta$ to $0° < \theta < 180°$ {and beyond}

h reprove the *Sine Rule* for acute angles $\angle A$ by inscribing triangle ABC in its circumcircle, drawing the diameter through B meeting the circle at A', and hence observing that $\frac{a}{\sin A} = 2R$, where R is the circumradius of triangle ABC; solve related problems

i show that in the 'ambiguous (ASS) case', the data may determine two possible triangles; link this observation with the RHS-congruence criterion

2D and 3D figures

a work freely with standard 2D figures (including right-angled, isosceles, and equilateral triangles; parallelograms, rhombuses, rectangles and trapezia; regular polygons; circles, circular sectors and segments, and shapes made of circular arcs)

b draw figures to scale; interpret distances, angles, and areas on maps and other scale drawings; apply similarity in analysing problems [e.g. derive the $\sqrt{2} : 1$ aspect ratio for the DIN A series of paper sizes]; understand how enlargement and scaling (or similarity) affects angles, lengths, areas, and volumes c talk about and work with common 3D shapes; find lengths and angles in 3D figures by considering 2D cross-sections; understand how to measure and to calculate the distance from a point to a line and from a point to a plane; understand how to calculate the angle between two planes; apply this to analyse familiar figures [e.g. use trigonometry or the *Cosine Rule* to calculate the angle between the base and the sloping faces in a square based pyramid, and between faces of a regular tetrahedron]

d calculate surface areas and volumes of cuboids and shapes made of cuboids, of 'wedges' (half a cuboid), of polygonal right prisms and cylinders, of cones and pyramids, of spheres and hemispheres; relate base radius and circumference, height, slant height, and surface area of cones

Circles

a understand and use the terms centre, radius, chord, diameter, circumference, tangent, arc, sector, segment

b understand the formula for the circumference of a circle and estimate π; calculate the length of the arc subtending a given angle at the centre; solve related problems

c relate the formula for the area of a circle to the formula for the circumference; calculate the area of a sector with given angle at the centre; solve related problems

d calculate the circumradius {and inradius} of a triangle with given data

e use *Pythagoras' Theorem* to find the equation of a circle of radius r—first with centre at the origin {then with centre at the point (c, d)}; find the points where a circle centred at the origin and a given straight line intersect {and the points where two circles intersect}; find the equation of the tangent to a given circle centred at the origin at a given point; relate this to the coordinates $(\cos\theta, \sin\theta)$ of the general point P on the unit circle with centre the origin and angle θ measured anticlockwise from the positive x-axis; {complete the square to identify certain quadratic equations as circles, and to find their centre and radius}

f {find the equation of the tangent to a given circle at a specified point; solve related problems}