

MATHEMATICS GCSE (ENGLAND)
PROPOSED SUBJECT CONTENT:
SUGGESTED REVISIONS. II

TONY GARDINER

The draft for *Consultation on GCSE subject content and assessment objectives*[†] was published on 11 June 2013 for comment by 20 August. This is the document that will determine the nature of assessment in England at age 16 for the next 10 years or so—but the draft still needs serious attention if it is to fulfil this function effectively. In a previous response *Suggested revisions. I*[‡] we indicated how to sharpen the first three content sections—**Number; Algebra; Ratio, proportion, and rates of change**. Part II focuses on the final three content sections: **Geometry and measures; Probability; Statistics**.

Given the timescale, one would prefer to keep suggested changes to a minimum in order to avoid wherever possible the need for serious redrafting. That is, at the ‘consultation’ stage the proposed draft should ideally need only minor ‘tweaking’ to help it to achieve its own declared goal of adopting a curriculum

- that builds upon the foundations that have been (partially) laid at earlier stages,
- that ensures progression to A level, and
- that is consistent with that of high-attaining education systems internationally.

In this regard, *Part I* was relatively straightforward. The debates of the last 30 years (stretching from CSMS around 1980 to the recent ICCAMS repeat study, with the evidence from repeated iterations of the National Curriculum and the National Strategies in between) have given rise to a degree of understanding in the wider mathematical community about number, ratio, and algebra. We may not agree on all the details, but there is a much greater awareness than there once was that competence in these domains is essential if students are to be able to use what they have been taught, and if those who wish to do so are to be in a position to make progress after the age of 16. *The same cannot be said when one comes to consider geometry and probability and statistics*, where we have yet to engage in a similarly serious professional debate, where strongly expressed beliefs are much less

2000 *Mathematics Subject Classification* 97A80, 97B70.

© 2013 Anthony D. Gardiner

[†]*Consultation on GCSE subject content and assessment objectives*, <http://www.education.gov.uk/schools/teachingandlearning/qualifications/gcses/a00225719/gsce-consultation>; *Mathematics GCSE subject content and assessment objectives* https://www.education.gov.uk/consultations/downloadableDocs/GCSEMathematics_final.pdf

[‡]A. D. Gardiner, *Mathematics GCSE (England). Proposed subject content: Suggested revisions. I*. The De Morgan Journal 3 (2013), 7–15. http://education.lms.ac.uk/wp-content/uploads/2012/02/Gardiner_GCSE_22June2013.pdf.

rooted in evidence, where politicians are more hesitant about interfering, and where the mathematical community has not yet given a clear lead.

Hence, the suggested changes in *Part I* and in *Part II* need to be understood differently.

- The suggested changes in *Part I* were designed to constitute clear improvements in wording and sequencing of the draft, but were also framed in a way that is intended to reflect a *shared understanding* in the wider community.
- The challenge presented by the three remaining content sections addressed in *Part II* is much more delicate. So the suggested changes to the draft, though extensive, have a more modest goal. We are in no position to suggest an ‘agreed’ coherent programme in these domains, and we cannot work towards convergence within the timeframe of the present consultation. Hence the aim here is *to remove as many infelicities as possible*, while providing a compromise wording that could serve both as a workable foundation for GCSE, and as a basis for continued professional discussion of the central issues.

How the themes of Part II (“geometry” and “probability and statistics”) differ from those of Part I—and from each other

The process whereby we internalise the ‘mental universe’ of *Number* and of *Algebra*, so that we can calculate freely with imagined numerical objects, is more subtle than the public and politicians tend to realise. Yet this process remains *relatively accessible*, in that many of the issues that concern the professional can also be explained to lay observers. Moreover, the *mathematisation* of number, of quantity and measures, and later of algebra, is a relatively *direct* process, which does not require the mastery of any intermediary discipline; so the step which the learner has to take from initial experience to the underlying mathematics is at first largely a matter of ‘assimilation’ (in Piaget’s parlance).

The situation in the teaching of *geometry* and *probability and statistics* is rather different. And while these domains have some common features, they also exhibit significant differences from each other.

Geometrical experience is a profound part of the infant’s visual, tactile, and experiential introduction to the world—perhaps more profound than the child’s experience of ‘numerosity’. Our experience of *variability* and uncertainty also begins very early. Thus both *Geometry* and *Probability and statistics* offer extensive opportunities for pre-mathematical work—exploring space, or collecting and making sense of data. But in both domains, the difficulties arise when one tries to move beyond this exploratory activity to develop an effective ‘calculus’ of formal mathematical calculation and deduction. In contrast to number and algebra, the step from pre-mathematical experience (of *space*, and of *variability*) to the mathematics of elementary *geometry* and *statistics* requires an acceptance of formalisms which involve the kind of conceptual discontinuity called ‘accommodation’ in Piaget’s framework. This does not mean that engagement with the material should be indefinitely delayed; but it does mean that serious thought is needed as to what effective ‘mathematisation’

one should be working towards at school level, and what preliminary foundations need to be laid if these more subtle arts of ‘mathematical calculation’ are to become a useful part of students’ education.

Geometry

Geometry has been taught for millennia, and the successes and failures of the last 2000+ years have much to teach us about how to ensure progress from pre-formal activity to formal calculation. The shift from vertices and edges of physical shapes first to points and lines drawn on paper, then to the abstract *points* and *lines* of formal geometry, is relatively unproblematic. The physical experience of building rigid 3D structures also supports the idea of SSS congruence and the strategy of analysing all configurations in terms of their constituent *triangles*. And the step from working with *particular* triangles to reasoning about a *general unknown* triangle ABC (that is, thinking about “the set of all possible triangles”—*represented in thought* by a single labelled triangle, about which we know nothing other than that it is a triangle) is surprisingly accessible.

Around 1900 there were extensive efforts to inject practical work, technical drawing, constructions, etc. as a preliminary stage in the teaching of geometry. But for a long time relatively few children stayed on at school; so the formalisation at age 14+ was still restricted to a minority and remained “traditional” in spirit. This led to the Euclidean approach being perceived as ‘elitist’. In the 1950s and 60s the relevance of the Euclidean approach was called into question—partly from a desire to be “more modern”, and partly from an increasing awareness that we had neglected the needs of the majority. Instead of looking for ways of refining the traditional approach, it was cast aside in favour of unproven alternatives. Dieudonné’s advocacy of *linear algebra* and *affine geometry* may have seemed logical to him, but it deprived the beginner of the fundamental psychological richness of Euclidean space. Attempts to link geometry with *motion* (through isometries, shears, enlargements, etc.), and to replace geometrical intuition by appeals to transformation groups and matrices, failed to recognise that the unifying strength of this approach only emerges *post hoc*, and that this made it unsuitable for use by anyone other than the very best students and teachers. Various ‘fun topics’ were also introduced, and some of these became fairly popular (rotations and reflections, patterns, nets, Euler’s formula, etc.).

Curricula and specifications drafted since 1980 have steadily diluted the ambitious goals of the 1960s, but have retained residual traces—so that we now have an incongruous collection of bits and pieces, none of which has delivered either a systematic ability to ‘think geometrically’, or an ability to apprehend and to analyse geometrical configurations in a mathematical way. There are references to bits of Euclidean geometry and ‘proof’ alongside allusions to ‘symmetry’ and ‘shapes’, with no apparent recognition that the result is an educational mess. For example, the most remarkable result in all of elementary geometry, *Pythagoras’ Theorem*, is routinely listed as something to be “applied” but not proved! And there is no attempt to specify the kind of framework (using SAS congruence and the basic property of parallel lines to prove the equality of triangles on the same base and between the same parallels) within which it is possible to give a suitable elementary proof.

There is no way this historical wasteland can be cleared and cultivated within the

timescale of the current consultation. So the suggested changes listed below merely seek to ameliorate some of the shortcomings in the draft in a way that might suffice as an outline GCSE subject content specification and at the same time allow subsequent professional debate. However, it would be dishonest not to declare that, my interpretation of the experience of the last 50 years is unambiguous: once one moves beyond the world of pre-mathematical experience, the only ‘geometrical calculus’ that has been shown to cultivate useable geometrical thinking in large numbers of adolescents—both those who are academically inclined and those of a more practical bent—is rooted in a parallel approach involving on the one hand drawing, construction and coordinates (analytic geometry), and on the other a simplified approach to elementary Euclidean geometry.

Probability and statistics

In contrast, the importance of *probability and statistics* for the modern world has dawned upon the public consciousness only relatively recently. Much interesting work has been done of a *pre-mathematical* kind. And while there appears to be a widely held belief that the mathematics of probability and statistics should have a central place in the school mathematics curriculum, there is rather little convincing evidence that we understand how this can be achieved, what the relevant prerequisites are, and at what age they are likely to be sufficiently mastered to make the material accessible. I recall being deeply impressed by the imaginative work in this direction undertaken by IOWO in the early 1980s (now the *Freudenthal Institute*); but I then discovered that colleagues in Dutch universities found that this approach seemed to have serious negative consequences for the competence of ordinary 18-year-old school leavers. In other words, there remains a worrying gap between aspiration and reality in this area—a gap which cannot be bridged by assertion, whether for or against the inclusion of such material. The truth would appear to be that *probability and statistics* are more subtle disciplines than other parts of elementary mathematics. Some of the reasons for this are explored on pages 43–45 of my *School mathematics curriculum for all written from a humane mathematical perspective*[†].

In short, while there is general agreement concerning the desirability of engaging with the mathematics of uncertainty, we are a long way from understanding how this can be achieved with ordinary pupils. What seems clear (though it is rarely openly acknowledged) is that any attempt to mathematise children’s primitive experience of uncertainty would seem to presuppose their achieving a level of mastery of fractions, ratio, and algebra that we in England have yet to attain. Hence much debate is still needed before one would dare to propose a specification likely to attract widespread approval. (My own tentative, and inexpert, efforts in this direction are available in the draft alluded to in the previous paragraph.) The list of suggested changes below seeks to do little more than to correct the most obvious infelicities, to remove what seems unrealistically ambitious, and to clarify what seemed to require clarification.

[†]A. D. Gardiner, A draft school mathematics curriculum for all written from a humane mathematical perspective: Key Stages 1–4, The De Morgan Journal 2 no. 3 (2012), 1–138. http://education.lms.ac.uk/wp-content/uploads/2012/02/KS_1-4_DMJ.pdf.

Suggested revisions (Part II)

[Roman type (for all) **and bold Roman (for a minority)** are copied verbatim from the consultation draft. *Italics* (for all) and ***bold italics*** (for a minority) are my suggested changes. Deletions are not indicated.]

“GCSE specifications in mathematics should require students to:

Geometry and measures

1. *understand and use standard units and related concepts (length, area, volume / capacity, mass, time, money, etc.)*
2. *derive and apply formulae to calculate plane areas (of triangles, parallelograms, trapezia, circles, and composite shapes), and surface areas, cross-sectional areas and volumes in 3-dimensions (of cuboids, polygonal prisms, cylinders, spheres, pyramids, cones, and composite figures)*
- 9a. *identify properties of, and describe the results of, rotations, reflections, translations, **glide reflections**, and enlargements (including with negative and fractional scale factors) applied to given figures in the coordinate plane*

[Item 10. In the draft is too vague and out of keeping with the other items; and it does not seem to have ever been openly debated. It could fit within a *different* coherent list, but does not seem to make sense in its present form and context; so it has been omitted.]

3. *calculate angles, lengths, and areas exactly in geometric figures, and approximately in settings represented by real or imagined maps and scale drawings; interpret the results of such calculations*
4. *use standard geometric language and notation (point, line segment, line, angle, vertex, side, parallel, perpendicular, plane, triangle, quadrilaterals and other polygons, regular polygons, etc.); use the standard conventions for labelling the sides and angles of triangle ABC; draw diagrams from written descriptions and use them to analyse imagined configurations*
- 7./9b./11a. *understand and use the basic congruence criteria for triangles (SSS, SAS, ASA. RHS); deduce and apply the consequent properties of parallelograms, and quadrilaterals; derive simple consequences for other plane figures (including regular polygons); understand that in the ambiguous ASS case the data may determine two possible triangles*
5. *derive and use the standard ruler and compass constructions (perpendicular bisector of a line segment, dropping/erecting a perpendicular to a given line from/at a given point, bisecting a given angle); **construct and work with the circumcentre/circumcircle of a given triangle**; recognise and use the perpendicular distance from a point to a line as the shortest distance to the line*
8. *apply the properties of angles at a point, angles at a point on a straight line, vertically opposite angles; understand and use the characterisation of parallel*

lines via alternate and corresponding angles; derive and use the sum of angles in a triangle (e.g. to deduce and use the angle sum in any polygon, and to derive properties of regular polygons); prove that the area of a parallelogram is equal to the area of a rectangle on the same base and between the same parallels, and deduce the standard formula for the area of a triangle

- 13./14. work freely with circles and related notions (centre, radius, diameter, chord, circumference, tangent, arc, sector, segment, etc.); **derive and use basic properties** (e.g. radius perpendicular to tangent; perpendicular bisector of chord passes through the centre; calculate arc length and area of circular sector from the angle at the centre and conversely); understand and use the fact that the tangents to a circle from an external point are equal
- 9c./11b. understand and use the notion of similarity and the basic criterion for similarity of triangles; know and use the effect of enlargement/similarity on angles, lengths, areas, and volumes
12. apply basic angle facts, congruence, parallels, and similarity to derive important consequences (including Pythagoras' Theorem), and use known results to obtain simple constructions and proofs

[It is sometimes claimed that 'proof' demonstrates that a conclusion is "true". It does not. Teachers need to know that *proof only shows that a conclusion "is true provided the hypotheses are true"*.

Some of the relevant hypotheses are openly declared within the statement of the result being proved: for example, in Pythagoras' Theorem, the triangle is assumed to be 'a right angled triangle'. But the *background hypotheses* which are implicit in the logical framework being used *are also part of the hypotheses*, even if they are taken as read.

It follows that **proof can only have meaning within a given logical framework**. And although the logical framework used in school is inevitably partly informal, the essential character of proof—and hence of mathematics—is lost if one authorises an eclectic, catch-all approach by suggesting one can "prove conjectures [...] using transformational, axiomatic, and property-based deductive reasoning".]

15. **derive the standard 'circle theorems' (angle subtended by a chord; angle between chord and tangent; angles of a cyclic quadrilateral and the converse; etc.) and use them to prove related results**
- 16./17./18. understand that trigonometric ratios for angles $< 90^\circ$ depend only on the angle; know the exact values of $\sin \theta$, $\cos \theta$, $\tan \theta$ for $\theta = 0^\circ, 30^\circ, 45^\circ, 60^\circ$; use trigonometric ratios to 'solve triangles' (that is, to find other angles and side lengths from given partial information); derive and use the identity $\sin^2 \theta + \cos^2 \theta = 1$; derive and apply the formula "area = $\frac{1}{2} \cdot ab \cdot \sin C$ "; **understand and use the Sine Rule and Cosine Rule**
6. apply familiar 2-dimensional ideas to analyse familiar 3-dimensional figures; work with 2-dimensional representations of 3-dimensional figures; find lengths and angles in 3-dimensional figures
- 19./20. **understand vectors as column vectors in 2-dimensions; interpret addition in terms of the parallelogram/triangle law; describe translations as 2D vectors; use multiplication of a vector by a (positive or negative) scalar; express a given vector in terms of other vectors;**

understand and use the modulus, or length, of a vector; derive an expression for the mid-point of a line segment; find the centroid of a triangle

[There is a significant lobby of experienced teachers who see vector work as fitting neatly at AS level (in **C1** where it currently reappears!), and who would argue strongly that students would be much better served at GCSE by more challenging work on analytic/coordinate geometry.

I failed to find anyone who could make sense at this level of item 21, so have omitted it.]

Probability

[The compromise wording suggested here is meant to be *enabling*. A detailed syllabus will need to specify what kind of “recording”, and what level of “analysis” is expected, etc.; but the evidence of the consultation draft is that we are not in a position to specify this kind of detail as part of the background GCSE subject content specification.]

1. *experiment*, record, and *analyse* the frequency of outcomes of *standard* probability experiments (*e.g. coin tossing and dice rolling*)

[*Standard* settings need to be carefully chosen so that their intrinsic structure means that they come equipped with an intuitive standard model (such as that ‘Head’ and ‘Tail’ are equally likely). This makes it possible at a later stage to extract the notion of a ‘probability space’ S which captures the totality of possible outcomes—each with its assigned numerical ‘probability’.

Despite considerable efforts I failed to discover a clear meaning for “frequency tree”, so the term is omitted.]

- 3./4./2. relate *observed* frequencies to theoretical probability; *know and use* the way probabilities *add, with total equal to 1*; *understand and use* ideas of randomness, fairness and equally likely events to *analyse* outcomes of *repeated* experiments (*including sequences of coin tosses and dice rolls, tossing two or more coins and rolling two or more dice*)
5. enumerate *simple events* and *compound events* systematically *working with* tables, grids, tree diagrams, and Venn diagrams; *use the sum and product rules for counting*
6. construct theoretical *probability spaces for simple examples related to the standard models of coin tossing and dice rolling*; *use these to calculate probabilities for compound events*
7. *enumerate and analyse possible outcomes* (including *using* tree diagrams and Venn diagrams *to decompose an event into the disjoint ways it can occur*); ***work implicitly with particular instances of conditional probabilities in the standard models***

[Most of the words in items 7–11. are rather hard to interpret at this level. The associated ideas—such as “independence”, “conditional probability”, “the formula $p(A \cap B) = p(A | B) \cdot p(B)$ ”, “theoretical distributions”, “risk”, “expected

outcomes”—are premature and hence widely misunderstood. The problems presented by their inclusion here are underlined by the fact that I was unable to be sure what the draft was trying to say. Whatever may have been intended these formal ideas are inappropriate for the GCSE subject content specification; instead GCSE should concentrate on preparing the ground so that important ideas can be refined and developed within post-16 courses for those who might need a more advanced treatment.

What is needed at this level is a framework within which students can be expected to engage with analyses and calculations that lay clear foundations in students’ experience of standard models for such things as the way probabilities add up, without necessarily at this stage imposing the general formula “ $p(A \cup B) = p(A) + p(B) - p(A \cap B)$ ”. The experiences gained will later lead to such notions as ‘*conditional probability*’, ‘*independence*’, ‘*expectation*’, etc., but these ideas first need some suitable soil in which to take root.

Some may argue that issues relating to risk and insurance (and even a qualitative discussion of the social and philosophical implications of the *Law of large Numbers*) deserve an airing in some form before the age of 16; but it seems unlikely that this can be sensibly located within the mathematics GCSE subject content specification.]

Statistics

1. *record and analyse the distribution of statistical data relating to a population; explore the statistics of ‘random samples’ taken from such populations, and understand that the ultimate objective is to relate the statistics observed in the sample to the statistics of the background population; consider what one can and cannot infer about properties of the hypothesised background population or distribution from an observed sample; work efficiently with the ‘average’ or mean of a set of numbers or measures; understand when other measures are more appropriate ways to summarise the population or sample ‘centre’*
2. *construct and interpret appropriate tables, charts, and diagrams, including frequency tables, bar charts, pie charts, and pictograms for categorical data, and vertical line (or bar) charts for ungrouped and grouped numerical data, and scatter graphs/diagrams for bivariate data*
3. *construct and interpret frequency tables, diagrams for discrete data and continuous data (including histograms with equal and unequal class intervals, and distributions); **construct and interpret cumulative frequency graphs***
4. *interpret, analyse, and compare the distributions of samples from univariate distributions through:*
 - *graphical representation (using discrete, continuous, and grouped data as appropriate)*
 - *appropriate measures of central tendency (mean and median; mode and modal class), spread (range, inter-quartile range), and cumulative frequency (range, quartiles and inter-quartile range)*

[The expression “univariate empirical distribution” used in the consultation draft does not appear to be standard—so I have had to guess that what was intended was to compare the observed distributions of two samples taken from a given background population.]

5. *explore* relationships in *sampled* bivariate data; sketch trend lines through scatter plots; *understand the intuitive, non-causal, notion of correlation*; **estimate lines of best fit**; make *and test* predictions; *consider the possibility* of interpolating and extrapolating *apparent* trends, *and the dangers of so doing*

About the author

TONY GARDINER (born 1947) is a British mathematician. He was responsible for the foundation of the *United Kingdom Mathematics Trust* in 1996, one of the UK's largest mathematics enrichment programs, initiating the Intermediate and Junior Mathematical Challenges, creating the *Problem Solving Journal* for secondary school students and organising numerous masterclasses, summer schools and educational conferences. Gardiner has contributed to many educational articles and internationally circulated educational pamphlets. As well as his involvement with mathematics education, Gardiner has also made contributions to the areas of infinite groups, finite groups, graph theory, and algebraic combinatorics.

In the year 1994–95, he received the Paul Erdős Award for his contributions to UK and international mathematical challenges and olympiads.

Email: anthony.d.gardiner>>>at<<<gmail.com