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LET'S GET REAL

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Number Systems to Order

The world of mathematics contains several different number systems. Today's children encounter these various structures in roughly the same order that their human ancestors discovered them. This process starts with the counting numbers $1, 2, 3, \ldots,$ or *natural numbers* as mathematicians like to call them. They typically then move on to the fractions or *rational numbers*—e.g.

$$\frac{2}{3}, \frac{7}{10}, \frac{15}{11}, \quad \text{etc.}$$

At some point negative numbers get thrown into the mix. Eventually the system of *real numbers* is arrived at (with a lucky few going on to discover the glories of the *complex numbers*).

This order

$$\text{natural} \longrightarrow \text{rational} \longrightarrow \text{real numbers}$$

might therefore seem an inevitable progression. In this post, we present a different view. Agreed, fractions may initially be easy to motivate: after all, the hunter-gatherer who had to divide one piece of fruit between two people certainly got the idea fairly rapidly. The trouble is that this approach to fractions, which essentially amounts to extended *counting*, soon becomes very fiddly.

For instance, when confronted with two whole numbers, only a small amount of knowledge is needed to judge which is larger. But comparing the sizes of fractions is far less straightforward. Quickly now, which is bigger,

$$\frac{3}{5} \text{ or } \frac{5}{8}?$$

Such questions can be answered, of course, but as a second step after the straightforwardness of the whole numbers, the methods involved may seem dauntingly technical.

But what we can we base our numerical work on, if not counting? The answer is *measuring*. As soon as humans needed to quantify distance, volume, weight, or time, they used numbers to do so. Today we know it is the system of real numbers which is right for the task. These are first met under the guise of *decimals*. The advantage of this presentation is that once place-value notation has been grasped, with whole numbers described via units, tens, and hundreds, the same line of thinking can be

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extended to address decimals of arbitrary length. With this done, it is almost as easy to discern the relative sizes of two decimals as it is for whole numbers.

All too often though, decimals are themselves introduced via fractions.

Measuring Numbers are Simpler than Counting Numbers

If, historically, humans encountered the rational numbers before the reals, there is another, more recent perspective from which we can see that the reals should perhaps come first. It is the observation made by Alfred Tarski in 1951 that the real numbers, are in a profound sense, simpler than the rationals. (A brief disclaimer: this is not intended to clinch the educational argument, but it is a pertinent and interesting body of knowledge which is not widely known.)

The work of Kurt Gödel, Alonzo Church, and Alan Turing had shocked the world by revealing that the system of integers is—far from being a simple tool used by cavemen—formally *undecidable*, and generally logically intractable. In 1949, Julia Robinson transported these insights into the rational realm ([**1**]). These too, she realised, form an undecidable, and astoundingly complicated mathematical structure.

But only two years later, Robinsons PhD supervisor Alfred Tarski swam against the prevailing tide of the times, to prove that the system of real numbers is, perhaps surprisingly, fully decidable. Whereas Gödel's theorem spoke of a deep mystery at the heart of arithmetic, if all you are interested in is measuring, the real numbers conceal no hidden horrors.

More precisely, the system of real numbers form what has recently come to be known as an *o-minimal structure* (see [**4**]). Roughly, this means that when comparing two real numbers (*qua* real numbers) all that really matters is their relative magnitudes. Most other facts that you might want to know follow from this. This is in striking contrast to the case of rational or natural numbers; when comparing two such numbers, one finds them surrounded by a mire of arithmetical considerations: which of our numbers (or their numerators or denominators) divide which others? Which are primes? Which are squares?

One does not have to dig deep for these questions to become intractable. But for most practical purposes they are also irrelevant: if I want to know if I have enough cash to buy something, I want to know how many dollars I have in my wallet. I don't care whether or not that number is prime.

Learning to Measure

Measuring doesn't require negative numbers, but it certainly demands numbers between the counting numbers. We call these numbers the positive reals; children might call them measuring numbers. They can think of them as good names for dots on positive number lines.

The commonest way to represent such a number is as an infinite decimal, although, sadly, we humans are usually forced to approximate them with a finite initial segment. We recommend thinking of measuring numbers as finite decimals that stand ready to be extended as far as necessary.

This leads to violating the strict rule that one of us had to follow in sixth grade: to read decimals as fractions. The teacher required reading 3.14 as "three and fourteen hundredths," although she herself probably didn't read either 3.1416 or 3.141592, both of which are better approximations of π, as fractions.

A great advantage to taking decimals one digit at a time is that preschool children can do it. Three-year-olds have little if any idea of what a hundred is, let alone a hundredth, yet any child who can recognize and say all the digits can read 3.141592 the way any mathematician would—which would *not* be as some number of millionths.

In the US, fractions, and the more easily understood decimals, are introduced earlier but are not studied in detail until middle school, where real numbers are usually (barely) introduced. Unfortunately, many middle school students—and adults—carry over inappropriate ideas about the ordering of whole numbers, after years of almost total focus on them, to real and rational numbers. One of us became interested in this phenomenon after seeing a video of a girl—who had just taken a calculus course!—saying that there were no numbers between 1.5 and 1.6 and then reading in [**3**] that fully a half of American and Canadian sixth graders thought that $0.5 < 0.12$. Did they also believe that $50 < 12$ or that $5 < 1.2$? We guess not.

In both the UK and the US, the number-line has become an essential educational tool, with both decimals and fractions introduced in the UK in Key Stage 2 (ages 7–11). If we start early, we believe children can use number lines and decimals to get a solid knowledge of the order properties of real numbers by the age of 10.

Let's stop rationalizing, admit that we are introducing real numbers *before* fractions, and focus our attention on doing it well. In the US, the National Science Foundation supported a Rational Number Project for twenty years. Is now the time for a two year Real Number Project?

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