

MATHEMATICS GCSE (ENGLAND)
PROPOSED SUBJECT CONTENT:
SUGGESTED REVISIONS. I

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The draft for *Consultation on GCSE subject content and assessment objectives*[†] was published on 11 June 2013 for comment by 20 August. This is the document that will determine the nature of assessment at age 16 for the next 10 years or so; and one suspects that even those who contributed to the drafting process might agree that, despite all the work that has been done, the current version still needs serious attention. Yet the timing of the consultation leaves the mathematical community insufficient time to analyse, to debate, and to reach a consensus on the changes that are needed. Moreover, July/August is scarcely the ideal time of year for hammering out a response—let alone a *coordinated* response.

There is no obvious ‘right way’ to respond to this situation. The art of drafting such national specifications is not something most of us find easy: we may discern curious omissions, or distortions, in the proposed version, yet not know how best to correct them. I hope I might therefore be excused for publishing *suggested revisions* as a contribution to discussion and debate.

To emphasise that the intention is constructive, I have accepted the constraint of working within the structure adopted in the draft listing of subject content, and suggest minimal changes and additions designed to improve the current version (rather than proposing what one might have preferred to see), in order to transform the draft into something more consistent with its own declared goal of adopting a curriculum

- that builds upon the foundations that have been (partially) laid at earlier stages,
- that ensures progression to A level, and
- that is consistent with that of high-attaining education systems internationally.

Compiling and cross-checking such “minimal changes and additions” is itself a major task, which should ideally take many months—time we do not have. In the hope of generating some communal debate and a degree of consensus it has proved necessary to publish this commentary in two parts, addressing the listed sections in groups. I appreciate that this is far from ideal (since each section impinges on other sections). But as Chesterton reminded us, in an emergency

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[†]*Consultation on GCSE subject content and assessment objectives*, <http://www.education.gov.uk/schools/teachingandlearning/qualifications/gcses/a00225719/gsce-consultation>; *Mathematics GCSE subject content and assessment objectives* https://www.education.gov.uk/consultations/downloadableDocs/GCSEMathematics_final.pdf

‘the best is often the enemy of the good’.

The goals of the consultation draft are quite different from those of the much more detailed *Mathematics curriculum for all written from a humane mathematical perspective*[†]; so the suggestions proposed here were initially drafted without consulting that much more detailed list, but were then modified slightly by checking against it.

Background

1. I think we should welcome the explicit recognition (page 3) that

“GCSE outcomes may reflect or build upon subject content which is typically taught at key stage 3 [KS3]”.

The spirit of *KS3 mathematics* for most students (whether completed by age 14, or revisited during key stage 4) remains *numerical* and relatively *concrete*:

- fraction arithmetic is mastered first in numerical form (rather than with algebraic fractions), where the entities have a definite magnitude, and answers can be easily interpreted;
- formulae may be used, but these formulae are quite different from abstract algebraic equations between meaningless unknowns; they are rather formulae which refer to tangible entities (such as the area of a triangle or circle) or which capture familiar relations (such as between speed, distance and time);
- the general principles underpinning ratio problems (e.g. the unit method) may be applied, but this is done in numerical, rather than purely algebraic, settings;
- powers are calculated, simplified, and interpreted, but they tend to be relatively familiar numerical powers, rather than algebraic powers;
- letters may be introduced systematically, and used in contexts where they have a specific meaning, but one hesitates before expecting students to simplify abstract compound expressions by applying the rules of algebra;
- angles, perimeters, areas and volumes may be calculated, simple deductions may be expected, and geometrical constructions implemented (bisecting segments and angles, constructing perpendiculars, etc.), but extended abstract geometrical proofs remain on the horizon;
- graphs may be drawn, but they are simple graphs that capture something that can be easily (and usefully) interpreted.

In short, there is a marked contrast between

- KS3 mathematics—where the foundations for subsequent abstract work are laid in relatively concrete, often numerical settings, and
- *KS4 mathematics* (for age 14–16)—where these foundations laid at KS3 in relatively concrete, often numerical, contexts are formulated and extended *in a way that involves abstraction at every turn*.

[†]A. D. Gardiner, *A draft school mathematics curriculum for all written from a humane mathematical perspective: Key Stages 1–4*, The De Morgan Journal 2 no.3 (2012), 1–138. http://education.lms.ac.uk/wp-content/uploads/2012/02/KS_1-4_DMJ.pdf

By the end of KS4, all students need confidence and competence in handling KS3 mathematics. For some students, this material will only need to be reinforced incidentally during KS4; but for others, it will need considerable strengthening beyond age 14 before students are in a position to proceed meaningfully and selectively to the abstraction implicit in more formal KS4 mathematics.

Hence GCSE assessment must be pitched at a level that ensures teachers are free to work towards this declared goal of helping **all** students achieve a genuine mastery of KS3 mathematics, while at the same time moving as many students as possible on to the more abstract KS4 material that is needed for the mathematics that is used in many subjects at higher levels.

*2. The consultation draft contains no discussion of assessment structures. However, the recognition that mastery of KS3 mathematics should be assessed **as a GCSE outcome**, and the clear consensus that there should be **two** GCSEs in mathematics (which we are repeatedly told need no longer pretend to be “equally demanding”) suggests:*

- (i) *a ‘higher’ GCSE which tests serious KS4 mathematics in a way that **prepares for A level studies**, and*
- (ii) *a ‘basic’ GCSE, which includes some elements of KS4 mathematics, but in which considerable weight is given to assessing genuine mastery of KS3 mathematics, and which is declared **not** to be a sufficient prerequisite for A level mathematics.*

It therefore makes sense to include limited, but explicit reference to introductory KS3 material within the GCSE specifications. This has the advantage of allowing each subsection to read in a more coherent way. (For example, the crucial subsection on “Ratio, etc.” in the consultation draft begins rather abruptly:

1. use ratio and scale factor notation, including $1 : r$

whereas the suggested revisions here include additional background that is needed by all students, and that should be an explicit part of any specification which reflects (ii) above.

3. In a similar spirit, one must welcome the principle adopted in the consultation draft of using **bold type** to identify aspects of KS4 content that are specifically intended for higher-achieving students (even if this target group is initially left undefined). The suggestions here include modest—but significant—changes to the material that is set in bold, or ordinary Roman, type. (For example, **all** students need to understand the effect of *repeated* percentage change, compound interest, and the effects of iterated growth and decay over time; any differentiation that is felt to be appropriate should be restricted to how the material is taught and assessed.)

Strategy

1. Given the time constraints, there would seem to be no way of agreeing on a different *structure* from that adopted in the consultation draft; so I have suppressed the temptation to improve the given framework of headings (Number; Algebra;

- Ratio, proportion and rates of change; Geometry and measures; Probability; Statistics).
2. Others may be better placed to assess whether the stated ‘Assessment Objectives’, and the associated percentages, need urgent adjustment, or whether their intended meaning can be clarified subsequently through enlightened interpretation (by the Awarding Bodies and by Ofqual).
 3. The ‘Appendix’ (also labelled ‘Appendix 1’ on page 12) is hard to construe. The list under (a) (pages 12–14) makes perfect sense in that these formulae should be known *by those who are expected to use them*. However, it is unclear what percentage of the cohort should be expected to understand, know and use the complete list. Hence the implications of the Appendix will depend on how the assessment issue raised in italics at the end of the previous section is handled. Knowing formulae makes perfect sense as long as the material being tested is pitched at a level that the target candidates are assumed to have *mastered*; thus a ‘basic’ GCSE should probably not require the complete list. Once this principle is understood, both the notion of a ‘formula sheet’, and the curious list under (b) (page 14), become redundant. (The definitions and formulae listed under (b) would appear to be exactly like those listed under (a)—and so should usually be known by target candidates who are presumed to have mastered the material which requires their use.)

The rest of these ‘*Suggested revisions: Part 1*’ concentrates on the first three listed sections: **Number**; **Algebra**; and **Ratio, proportion, and rates of change**. To make it easier for the reader to compare this version with the consultation draft I have retained the statement numbering in the draft, even where the ordering has been modified. I have also avoided trying to propose minor improvements at this stage, and so have kept the exact wording wherever this seemed feasible: in particular, Roman type (and **bold Roman type**) is copied verbatim from the draft (with a few words or expressions deleted). Where text has been changed or added, the changes are *in italics* (or **bold italics**).

Suggested revisions

[Roman type (for all) and **bold Roman (for a minority)** are copied verbatim from the consultation draft. *Italics* (for all) and **bold italics** (for a minority) are my suggested changes. Deletions are not indicated.]

“GCSE SPECIFICATIONS IN MATHEMATICS SHOULD REQUIRE STUDENTS TO:

Number

- 1a. *order positive and negative integers, decimals, and fractions; use the symbols $<$, $>$, \leq , \geq*
1. *apply the four operations, including formal written methods, to integers, decimals, and simple fractions (proper and improper)—both positive and negative; understand and use place value (e.g. when working with very large or very small numbers, and when calculating with decimals)*

2. *recognise and use relationships between operations, including inverse operations, to simplify calculations and expressions; use conventional notation for priority of operations, including brackets, powers, roots and reciprocals*
- 3a. *know and work with squares, cubes, square roots, **and cube roots**; recognise powers of 2, 3, 4, 5; extend to powers **and** roots of fractions and decimals; **estimate powers and roots of any given positive number***
- 3b. *know and work with standard integer sequences, generating terms via a position-to-term rule and via a term-to-term recurrence*
3. *calculate with roots, and with integer (positive, negative, and zero) and fractional indices; understand and use index laws in working with numbers*
4. *calculate exactly with fractions, surds, and multiples of π ; simplify surd expressions involving squares (e.g. $\sqrt{12}$) and **rationalise denominators***
5. *calculate with and interpret standard form $A \times 10^n$, where $1 \leq A < 10$ and n is an integer*
6. *work interchangeably with ‘decimal’ fractions (such as $\frac{7}{2}$ or $\frac{3}{8}$) and the corresponding terminating decimals; **change any given fraction into its (recurring decimal), and interpret a given recurring decimal as a fraction***
7. *identify and work with fractions in ratio problems*
 [The rewording in 6. and 7. replaces the opaque expression “apply relationships between” used in the draft.]
8. *define percentage as ‘number of parts per hundred’; interpret percentages and percentage changes as fractions or decimals, and interpret these multiplicatively; express one quantity as a fraction of, and as a percentage of, another; compare two quantities using percentages; work with percentages greater than 100%; apply repeated percentage change; solve reverse percentage problems*
9. *interpret fractions and percentages as operators (e.g. $\frac{3}{5}$ or 60% of a given quantity)*
10. *estimate answers; check calculations using approximation and estimation, including answers obtained using technology*
11. *round numbers and measures to an appropriate degree of accuracy (e.g. to a specified number of decimal places or significant figures); use inequality notation to specify simple error intervals due to truncation or rounding*
12. *apply and interpret limits of accuracy, **including upper and lower bounds***
13. *use the concepts and vocabulary of prime numbers, factors (divisors), multiples, common factors, common multiples, highest common factor, and lowest common multiple; factorise integers, calculate with integers in factorised form, understand and use prime factorisation*
14. *solve problems involving simple counting; **understand and use the product rule for counting***
15. ***multiply powers of a fixed base ‘a’ by adding exponents; introduce and use logarithms in base 2 and base 10; interpret exponential and log as inverse functions***

Algebra

[This is a key section—but one where the draft needs a lot more work. The current weakness is partly due to the failure to distinguish clearly be-

tween ‘expressions’, ‘formulae’, ‘equations’, ‘identities’, and ‘functions’, and the belief that algebra somehow becomes magically easier if one uses imprecise (often geometrical and graphical) language in an informal way: one does not ‘solve a formula’; one cannot ‘solve graphically’; one cannot ‘plot an equation’; one cannot easily ‘solve a linear equation in one variable approximately by using a graph’; etc.

A curriculum or specification needs to make a clear distinction by reserving the word “solve” for “*exact* solution”. It must therefore use different language for geometrical or graphical *interpretations*: these are important, but the suggestion that they can be used as a *substitute* for ‘solving’ is like confusing the *exactness* of Pythagoras’ Theorem with such crude methods as “finding the length of the hypotenuse” by drawing a scale diagram and measuring on the diagram.

The situation is further confused by the attempt to blur the distinction between *elementary algebra* (which is needed by, and accessible to, **all** students) and the language of *functions* (which becomes natural somewhat later, after extended successful engagement with elementary algebra). This approach would seem to restrict access to algebra for the majority of students, and evidently makes it harder to draft GCSE specifications without introducing misleading infelicities.]

1. *use and interpret algebraic notation (including ab in place of $a \times b$; $3y$ in place of $y + y + y$ and $3 \times y$; a^2 , a^3 , a^2b in place of $a \times a$, $a \times a \times a$, $a \times a \times b$; $\frac{a}{b}$ in place of $a \div b$, and with coefficients written as fractions rather than as decimals; etc.)*
 - 1a. *substitute numerical values into formulae and expressions and identify inadmissible values where appropriate*
 2. **manipulate algebraic expressions (including those involving surds and algebraic fractions) by**
 - * collecting like terms
 - * multiplying a single term over a bracket
 - * taking out common factors
 - * expanding products of two or more binomials
 - * *simplifying linear and quadratic expressions;*
 - * *simplifying single terms and more complicated expressions using the index laws*
 - * *factorising expressions, including quadratics with real roots; use the difference of two squares*
 - 2a. *understand and use standard mathematical formulae; rearrange formulae to ‘change the subject’*
 - 2b. *find x given particular values of y in simple equations (e.g. $y = \frac{k}{x}$, $y = \frac{k}{x^2}$)*
 8. *work reliably with coordinates in all four quadrants*
 3. *understand the difference between an equation and an identity; explain mathematically why two algebraic expressions are equivalent; use algebra to support and construct calculations and proofs; decide whether two given expressions are identical or not—then explain why they are, or show that they are not*

18. construct linear equations in one variable; *solve the general linear equation in one unknown* (including all *forms* that *first* require rearrangement)
 [This needs to be signposted as psychologically prior to the following expansion of 9. from the consultation draft. Only later will one be in a position to interpret the solution of $ax = b$ by setting $y = b$, or by thinking about the point where the line “ $y = ax$ ” meets the ordinate $y = b$, to get the particular point $(\frac{b}{a}, b)$ on the line “ $y = ax$ ”.]
9. set up a *single* linear equation in two unknowns; *interpret the equation as a constraint specifying the set of all points (x, y) that satisfy the equation—that is, the set of points lying on a straight line in the coordinate plane; relate the gradient and position of the line to the coefficients of the equation; understand and use the relation between gradients of parallel and perpendicular lines; establish the link between gradient and ‘rate of change’ of y with x ; find the equation of the line through two given points, or through one given point with given gradient*
- 23a. solve **any pair of** simultaneous linear equations in two variables by *eliminating a variable; interpret the analytic solution as ‘finding the point of intersection’ (if any) of the two lines*
4. *translate simple situations or procedures into algebraic expressions or formulae; use this to solve problems where one can derive an equation (or two simultaneous equations), solve the equation(s) and interpret the solution*
20. *factorise quadratic expressions $ax^2 + bx + c$ in one variable; solve quadratic equations that reduce to $ax^2 + bx + c = 0$ (including those that require rearrangement) by factorising; interpret $y = ax^2 + bx + c$ as a constraint specifying the set of all points (x, y) that satisfy the equation—that is, the set of points lying on the parabola; interpret the solutions of the equation $ax^2 + bx + c = 0$ as those points where the parabola crosses the x -axis*
- 20a. *use the known expansion of $(x + a)^2$ to solve the general quadratic by completing the square; deduce the symmetry of the graph of $y = ax^2 + bx + c$ about $x = -\frac{b}{2a}$, and sketch the general quadratic, 21. **identifying the turning point:** understand and use the quadratic formula; solve equations involving rational expressions that reduce to a quadratic*
- 20b. *recognise and use the equation of a circle with centre at the origin; **find the centre and radius, and sketch the curve, in simple cases where the centre of the circle is not at the origin; given a circle, find the equation of the tangent at a given point***
22. **explore simple ways of finding approximate solutions numerically via iteration**
23. construct and solve *two* simultaneous equations in two variables, *where one or both may be quadratic; interpret the result geometrically*
24. solve linear **and quadratic** inequalities in one or *two* variables; represent the solution set on a number line *or in the plane, and using set notation*
14. recognise and use *the sequences of squares, cubes, prime numbers, simple arithmetic progressions, triangular numbers, and particular geometric progressions (including $2^n, 3^n, 10^n, (\frac{1}{2})^n, (\frac{1}{3})^n, (\frac{1}{10})^n$)*
 [It is inappropriate to confuse the initial study of *sequences* and growth by including “the *sum* of an arithmetic *series*”.]

13. generate the terms of a sequence
 (a) using a *given* term-to-term rule,
 (b) using a position-to-term rule (including examples where the n th term is defined by a configuration depending on n);
15. find simple position-to-term rules (i.e. a ‘closed formula’) that give rise to given sequences (including examples where the n th term is given by a linear or quadratic expression); find the position-to-term rule implied by a given term-to-term rule
 [One cannot “deduce linear and quadratic expressions to calculate the n th term”.]
- 14a. understand that when $x < 1$ (**or** $|x| < 1$) the sequence (x^n) of powers of x tends rapidly to 0, and that when $x > 1$ (**or** $|x| > 1$) the sequence (x^n) of powers of x grows rapidly without bound; make the connection to compound interest, to population growth, to doubling times, and to radioactive half-life;
17. **explore linear, polynomial, and exponential sequences (including Fibonacci) arising in mathematics and in other contexts**
25. interpret the equations $y = mx + c$ and $y = ax^2 + bx + c$ and their graphs as ‘functions’; sketch the graph of any quadratic function by completing the square; explore applications (**including: distance-time graphs for uniform motion and speed-time graphs for uniformly accelerated motion; distance-speed and distance-time graphs for uniformly accelerated motion**)
- 25a. **plot and interpret graphs of reciprocal functions $y = \frac{k}{x}$, simple cubic functions, particular exponential functions $y = k^x$ for easy positive values of k ; extend the trigonometric ratios for angles less than 90° to the circular functions \sin , \cos , \tan (with arguments in degrees) and their graphs**
 [There are good reasons to include a serious treatment of the modulus function (at least in bold type). But it is hard to see why the draft omits this, yet includes “piece-wise linear” functions.]
26. **calculate or estimate the gradient of a simple quadratic curve at a given point, and the area under a graph of rates (including velocity-time graphs) and interpret the results**

Ratio, proportion, and rates of change

- 0a. change freely between related standard units (time, length, area, volume/capacity, mass, etc.) and compound units (speed, rates of pay, prices, density, pressure, etc.) in both numerical and algebraic contexts
- 0b. find and recognise fractional parts of shapes and quantities; express one quantity as a fraction of another
- 0c. identify, and find required properties in, similar figures; use scale diagrams and maps, and interpret scale factors; understand and use the effect of scaling on different quantities (including angles, lengths, areas, and volumes)
1. divide a given quantity into two parts in a given part-to-part, or part-to-whole ratio; use ratio and scale factor notation $m : n$ and $1 : r$, initially where m , n are positive integers and r is rational; extend usage to $a : b$ and $1 : c$ where a , b , c are real (e.g. $\sin 45^\circ$); express the division of a quantity into two parts as

a ratio; work with separate quantities in a given (external) ratio; reduce a ratio to its simplest form; apply these ideas to solve standard problems

[This is one place where I have omitted an extended sequence of words

“apply methods involving conversion, mixing, measuring, scaling, comparing quantities and concentration”

because

- (a) in compressing too many ideas the meaning became opaque,
- (b) I suspect the intended ideas are covered by proposed additions elsewhere.]

- 2a. *understand and use ‘proportion’ as ‘equality of ratios’*
2. compare lengths, areas, and volumes using ratio notation; make links to similarity (*including trigonometric ratios*) and scale factors
- 2b. *relate the language of ratios and the associated calculations to the arithmetic of fractions*
3. construct and interpret equations that describe direct and inverse proportion; *understand ‘X is inversely proportional to Y’ as ‘X is directly proportional to $\frac{1}{Y}$ ’*
4. *interpret the gradient of a straight line graph as a rate of change*; recognise and interpret graphs that illustrate direct and inverse proportion
5. *solve proportion problems involving speed, rates, and other compound units*
6. **interpret the gradient at a point on a curve as the *instantaneous* rate of change; apply the concepts of instantaneous and average rate of change (*gradients of tangents and chords*) in *numerical, algebraic and graphical contexts***
7. *interpret percentage change as a ratio of quantities (before and after); understand and work with simple interest and the effect of repeated growth*
8. *set up, solve, and interpret the answers in growth and decay problems, including compound interest; **work with more general iterative processes***
9. *use the ideas and language of ratio, direct and inverse proportion, and rates of change in algebraic, graphical, and geometric contexts*

About the author

TONY GARDINER (born 1947) is a British mathematician. He was responsible for the foundation of the *United Kingdom Mathematics Trust* in 1996, one of the UK’s largest mathematics enrichment programs, initiating the Intermediate and Junior Mathematical Challenges, creating the *Problem Solving Journal* for secondary school students and organising numerous masterclasses, summer schools and educational conferences. Gardiner has contributed to many educational articles and internationally circulated educational pamphlets. As well as his involvement with mathematics education, Gardiner has also made contributions to the areas of infinite groups, finite groups, graph theory, and algebraic combinatorics.

In the year 1994–95, he received the Paul Erdős Award for his contributions to UK and international mathematical challenges and olympiads.

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