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NATIONAL CURRICULUM (ENGLAND), SEPTEMBER 2013; ATTAINMENT TARGETS AND PROGRAMMES OF STUDY (KEY STAGES 1–3). COMMENTS AND SUGGESTED NECESSARY CHANGES

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The *Education Order 2013*[†] was "made" on 5 September 2013. The relevant details were "laid before parliament" on 11 September 2013, and will come into effect on 1 September 2014. Some of the details for GCSE were published on 1 November 2013. Further elaboration of GCSE assessment structure, and curriculum guidance for Key Stage 4 (Years 10–11, ages 14–16) are awaited.

It is generally agreed that the curriculum review process adopted over the last 3–4 years has been seriously flawed. Those involved worked hard, often under very difficult conditions. But the overall approach (of relying on civil servants and drafters whose responsibilities and constraints remained inscrutable) has merely demonstrated that drafting and maintaining curricula is a *specialist* task, requiring dedicated professionals with *specialist* experience.

Whatever flaws there may have been in the process, we will all have to live with the new curriculum for some years. So it is important to have an open discussion of the likely difficulties. This article is an attempt to indicate aspects of the mathematics programmes of study [‡] that will need to be handled with considerable care, and revised in the light of experience.

After three years of widespread unease about the *process* of the curriculum review and its apparent *direction*, it is remarkable that *there has been almost no media coverage, and no clear professional response* to the final mathematics programmes of study for ages 5–14. There is therefore a real danger that insights that emerged along the way will simply be forgotten, and that the same mistakes may then be made next time. Some of these 'insights' focused on the details of curriculum design (e.g. my own essay[§]). Others have taken a broader view (e.g. Paul Andrews' 2012 MA Presidential Address[¶]).

The details laid before parliament are 'statutory'; but they incorporate basic flaws,

2010 *Mathematics Subject Classification* 97B70.

[†]The Education (National Curriculum)(Attainment Targets and Programmes of Study)(England) Order 2013 No. 2232. `http://www.legislation.gov.uk/uksi/2013/2232/pdfs/uksi_20132232_en.pdf`.

[‡]National curriculum in England: mathematics programmes of study. `https://www.gov.uk/government/publications/national-curriculum-in-england-mathematics-programmes-of-study`.

[§]A. D. Gardiner, *A draft school mathematics curriculum for all written from a humane mathematical perspective: Key Stage 1–4*, De Morgan Journal 2 No.3 (2012) 1–138.

[¶]P. Andrews, *Learning from others. Can PISA and TIMSS really inform curriculum developments in mathematics?*, Math. Gazette 96 (537) (2012), 386–407.

and significant contradictions between the statutory list of content (which could all-too-easily be imposed uncritically) and the declared over-arching "aims" (which could get forgotten, or ignored). Given these flaws, the fate of the new programmes of study will depend on how sensitively their implementation is handled—whether slavishly, or intelligently. Teachers—and Ofsted, senior management, etc.—need to be alert to those aspects of the stated programmes of study that incorporate predictable pitfalls.

We summarise here what seem to be the two most important flaws.

> Some material in Key Stage 1 and 2 is very poorly specified (especially from Year 4 onwards).
>
> Some items are listed *unnecessarily and unrealistically early*, and so may be introduced at a stage:
> (a) where they are not yet needed,
> (b) where they will not be understood,
> (c) where they will be badly taught, and
> (d) *where—if the relevant requirements were relaxed—the premature material could easily be delayed without causing any subsequent problems.*

The listing of material prematurely may have reflected a Ministerial desire to 'raise aspirations'. But such good intentions will not reduce the consequent damage. Hence if there is a consensus that certain expectations are excessive or premature, then this needs to be admitted so that teachers will feel free to focus on what really matters, and to work towards an *intelligent interpretation* of the listed material. This article is offered as a guide to those (in Ofsted, in senior management in all schools, mathematics coordinators in primary schools, heads of department in secondary schools, etc.) on whom we all depend for a sensible interpretation of the detailed mathematics programme of study.

Key Stage 3 has been a source of weakness in England for some time. The last two Ofsted triennial reports[†] revealed the extent of current weakness in mathematics teaching at Key Stage 3. And the results of independent studies (such as TIMSS 2003, 2007, and 2011[‡], and ICCAMS) have strongly reinforced this assessment.

> The listing of content for **Key Stage 3** is in some ways reasonable, but too many things are left implicit. The programme of study is less structured than, and contains less detail than, that for Key Stages 1

[†]Ofsted, Mathematics: understanding the score, 19 Sep 2008, `http://www.ofsted.gov.uk/resources/mathematics-understanding-score` and Mathematics: made to measure, 22 May (2012), `http://www.ofsted.gov.uk/resources/mathematics-made-measure`.

[‡]TIMSS and PIRLS, `timss.bc.edu`.

and 2. Hence the details of the Key Stage 3 programme need interpretation. At present:
(a) the words of each bullet point are rarely elaborated;
(b) the connections between themes are mostly suppressed; and
(c) there is no mention of essential preliminaries.
In addition
the Key Stage 3 programme has no accompanying 'Notes and guidance'.

In summary, if the declared goals for Key Stage 4 are to be realised,

- we need some way of clarifying the specified content and relaxing the unnecessary and potentially damaging pressures built in to the **Key Stage 1–2** curriculum as it stands; and
- the centrally prescribed curriculum for **Key Stage 3** needs to be much more clearly structured to help schools understand what it is that is currently missing at this level—initially *by providing suitable non-statutory 'Notes and guidance'*.

After some general comments in Section 1, I shall work through the programmes of study document line-by-line, but with a relatively light touch—mostly avoiding minor quibbles and focusing on what is most striking. Section 2 makes some general remarks on the tension between the content listed in the detailed programme of study and the declared 'Aims'. Section 3 comments on the detail in the preamble ('Purpose of study', 'Aims', etc.). Finally Sections 4–6 go through the listed content in the 'Programmes of Study' for Key Stages 1–3 in turn, highlighting apparent errors, pointing out confusions, and commenting on features which seem worthy of immediate comment.

As with all 'paper exercises', this initial trawl may misconstrue parts of the document, or overlook important details. So, in the spirit of *The De Morgan Forum*†, readers are encouraged to respond with their own comments, criticisms, etc..

1. *Background*

1.1. It is precisely the proven importance over many centuries of key mathematical content that justifies the central position of mathematics in the curriculum. The decision to review the curriculum derived in part from the view that the purpose of *pedagogy* is to convey this key *content* more effectively. Hence, insofar as certain pedagogical approaches have failed to achieve *competence* for many pupils in key content areas, these approaches deserve to be challenged.

However, there is a danger that the new curriculum may suffer from the opposite weakness: namely that it constitutes a poorly organised list of content divorced from the kind of supporting didactical *framework* that is needed if teachers and pupils are to see that the enhanced content demands are in fact achievable. We therefore need to find some way of ensuring

† *The De Morgan Forum*, `education.lms.ac.uk`.

(a) that teachers are discouraged from simply presenting the material as listed, without first 'preparing the soil' in which the central ideas can take root,
(b) that teachers are supported in making the necessary connections. and
(c) that teachers do **not** feel pressured to 'cover' important content prematurely, or to do so in such a superficial way that competence for most pupils will remain elusive.

1.2. Incoming Ministers in 2010 had concerns about school mathematics, which persuaded them that the *status quo* deserved to be challenged. Many of their concerns were genuine. But the evidence on which these concerns were based was partly misconstrued.

- *At upper primary level*, teachers achieved impressive rises in 'average scores' on TIMSS in 2003 and 2007—rises which were sustained in 2011. This success needs to be recognised. The hidden weakness lies in the fact that these were *average scores*. And average scores tend to reflect performance on the bulk of relatively simple tasks (such as 'getting answers' to simple calculations, where the mark scheme pays no attention to *the efficiency or inefficiency of the methods used*). A closer analysis shows that English pupils perform very poorly on items which require slightly more than simple calculation, and which depend on efficient use of the kind of 'structural arithmetic' that serves as preparation for *subsequent progress at Key Stage 3*.
- *At lower secondary level*, current weakness in England is *more serious* than is generally acknowledged. Dramatic 'improvements' in domestic GCSE results outran the stagnant *average* scores on external measures (such as TIMSS). There remained a possibility that there might still have been a modest degree of improvement at age 14–15 since 1995. However, a more careful study of test items (e.g.

    (a) on TIMSS, where the marginal rise in 2007 was partly reversed in 2011; and
    (b) on ICCAMS, which repeated test items for 14–15 year olds from 30 years ago)

  shows that, while English Year 9 pupils may score tolerably on certain simple tasks, they perform consistently poorly—and often worse than 30 years ago—on tasks which are *diagnostic of preparedness for subsequent study.*

Short-term accountability measures would appear to have led many teachers to reinforce crude and inflexible ways of 'getting answers'—since they see this as a simple way of securing the objective of meeting the externally imposed requirement of improved results on the next test. In contrast, medium term progress in mathematics depends on achieving competence in *familiar* tasks, while doing this in ways that lay the foundations for using *less familiar*, 'forward-looking' methods, whose pay-off will come *at the **next** key stage*. That is, each key stage has become myopic—being satisfied with getting answers to simple tasks *using primitive methods*, and failing to recognise that the central function of each key stage should be to 'move pupils on', and to lay the conceptual foundations that are needed for *subsequent* development: that is, *to prepare for what lies ahead.*

1.3. Thus one missing ingredient in the new programme of study is its failure to emphasise the contrast between

(a) **backward-looking methods** that get answers in the short-term, but which trap pupils in old ways of working (as with finger counting, or idiosyncratic calculation methods, or reducing multiplication to repeated addition, or modelling questions about fractions in terms of pizzas—all of which may have transitional value, but which are known to block later progress if they become too strongly embedded), and
(b) **forward-looking methods**, that may seem unnecessary if the perceived goal is merely to get answers to simple problems *at a given stage*, but which are important because they reflect the inner structure of elementary mathematics and are essential for *progress at the next stage*.

It is not easy for a mere listing of curriculum *content* to capture this crucial distinction which underpins all effective mathematics teaching. And insofar as the revised programme of study incorporates this idea, it tends to do so in ways that are not immediately apparent.

1.4. The listing of content in the programme of study is also rather dry and 'formal'—focusing on 'what' rather than 'how'. In one sense, this emphasis is healthy. But it ignores the essential interplay between content and *didactics*. To serve as a useful guide a content list, or programme of study, needs to be constructed in a way that indicates, and supports, a clear underlying 'didactical architecture'. In contrast, the given programme of study often fails to convey not only key central principles (such as the contrast between *backward-looking* and *forward-looking methods*), but also important details (such as the key didactical stages which can lead from talking about "half a pint" or "half an hour" in Year 2 to competence with fractions in Year 9). Procedural fluency is rightly stressed. But this is too often stressed in isolation, as though a robust grasp of place value, for example, will emerge spontaneously simply as a result of banging on about fluency in specified procedures. It won't; so something more is needed.

2. *The need for flexibility*

2.1. The document as a whole, and the programme of study for each key stage, begin with certain important general statements. However the detailed content listing needs to be re-structured and re-worded to reflect these important statements.

The three overall 'Aims' correspond to the three 'Assessment Objectives' for GCSE, and simply assert that

> "all pupils become **fluent, reason mathematically**, and can **solve problems**".

The 'Aims' also include a clear statement that pupils should "make connections across mathematical ideas". It is true that the listing of content contains repeated emphasis on *fluency*—with mathematics consistently presented as a string of new things to be taught and mastered. But there are few signs of pupils being required to explain, to justify their answers, or to make specified connections. In short, opportunities to cultivate *mathematical reasoning* are routinely overlooked: to give just two examples: when multiplication is mentioned in Years 1 and 2, there is no mention of 'repeated addition'; and the link between measures and decimals is poorly formulated throughout. The expression "solve problems" is used fairly frequently, but it comes across as yet another burdensome item in the list of things to be 'covered', rather than as an important and challenging educational activity with its own didactics.

On the opening page of the programme of study for mathematics we read (under 'Aims'):

> "***decisions about when to progress should always be based on the security of pupils' understanding and their readiness to progress to the next stage.***"

Yet there is no recognition (here or elsewhere) of the tension between this excellent general principle and the assertion on the very next page (under 'Attainment targets') that:

> "*By the end of each key stage, pupils are expected to know, apply and understand the matters, skills and processes specified in the relevant programme of study.*"

The only declared flexibility in what is "expected" is the assertion (under 'School curriculum') that:

> "*Schools are [...] only required to teach the relevant programme of study by the end of the key stage. Within each key stage, schools [...] have the flexibility to introduce content earlier or later than set out in the [year-by-year] programme of study.*"

The stated flexibility "*within* each key stage" may prove sufficient to allow teachers to adjust for some of the over-ambitious goals. For example, it is stated that all pupils should know their multiplication tables by the end of Year 4. The goal itself is now generally welcomed; but there appears to be a clear consensus that Year 5/6 will prove more realistic for many pupils. However, it is **only in such instances**—where the delay is **within** a key stage—that teachers are free to appeal to the above 'aim' in adjusting their expectations. Where a theme which is listed in Year 2 (i.e. **the end** of Key Stage 1), or in Years 5 and 6 (i.e. **the end** of Key Stage 2), also proves to be premature for many pupils, teachers are given no such flexibility. In some cases the over-optimism is implicitly acknowledged by including these premature themes again at the start of the next key stage. Yet, as things stand, teachers in the final year of each key stage are not free to prioritise their pupils' *long-term* progress, and to base their

> "decisions about when to progress ... on the security of pupils' understanding".

We will therefore need some *ad hoc* way of adjusting the statutory expectations placed on teachers and pupils to allow some material that is currently listed in the *final* year of a key stage (i.e. in Year 2, or in Year 6) to be largely deferred to the *next* key stage (i.e. in Year 3, or in Year 7). In other words, the statutory document needs to be interpreted *imaginatively* rather than literally.

In short, teachers will need the confidence, and the necessary support (from their head teachers, from Ofsted, etc.) to *interpret* the given programme of study intelligently, and to focus attention at each stage on those themes that will allow pupils to master as much of the prescribed programme of study as possible, **while always respecting the above fundamental aim** (in bold).

This need for an imaginative interpretation will be especially pressing in primary schools, where the general consensus is that the specified progra

- the necessary prerequisites will not be in place,
- some of the material is not needed at that stage, and
- if schools are nevertheless forced to address this material, the result is likely to be the opposite of the declared 'Aim'—namely

> "**in**security of understanding, and **the opposite of readiness** to progress".

2.2. The need for teachers to exercise judgement flexibly, which was discussed in 2.1, is further elaborated in the 'Introduction' to the Key Stage 3 programme, where we read:

> "The programme of study [. . .] is organised into apparently distinct domains, but pupils should build on key stage 2 and connections across mathematical ideas to develop fluency, mathematical reasoning and competence in solving increasingly sophisticated problems. [. . .] Decisions about progression should be based on the security of pupils' understanding and their readiness to progress to the next stage. [. . .] Those who are not sufficiently fluent should consolidate their understanding [. . .] before moving on."

Unfortunately the "connections across mathematical ideas" are largely invisible in the ensuing list of Key Stage 3 content. And while 'fluency' is a valid diagnostic, the repeated emphasis on 'fluency' and 'practise'—and the fact that the intended meaning of these words is never clarified—could all-too-easily result in a regime of unimaginative repetitive work that will not deliver what is wanted.

So in making the best of the statutory curriculum, it is essential for us all to remember the declared 'Aims'. And teachers will have to be bold in insisting on their responsibility to make

> "decisions about when to progress based on the security of pupils' understanding and their readiness to progress to the next stage."

Only then can we hope to identify and correct infelicities in the current document, and work together not only to agree on higher aspirations, but also to reach a new consensus as to how these aspirations can be achieved.

3. *National curriculum in England: Purpose of study, etc.*

Aims. The overall aims are reasonably well expressed—but still need some improvement. For example:

(a) We are rightly told that pupils should make "connections". However, the "connections" that are currently neglected are often very basic—and for these the word "rich" is a distraction which will make it harder for teachers to understand exactly what is being referred to.

(b) This concern is exacerbated by the way the programme of study consistently misses opportunities to explain exactly what these missing "connections" are.

(c) The suggestion that the scope for challenge should be exhausted "before any acceleration" is most welcome. But the aims mislead slightly when they suggest that the focus on challenge should be "offered through *rich and sophisticated* problems"—since we know (for example from TIMSS and ICCAMS) that our very best students currently struggle when they are faced with the *simplest two-step* problems.

Spoken language. What is stated about spoken language is welcome. However, the general silence about *written* language is regrettable, as is the fact that, when the silence is broken, it is too often broken in a painfully blinkered way: for example, we find in several places:

> "Pupils should read, spell and pronounce mathematical vocabulary correctly".

There is no evident recognition of the need for pupils to go beyond vocabulary and to learn to read and interpret text, diagrams, numbers and other symbols, and information given in problems. Nor is sufficient attention given to the profound link between mathematics instruction and language (both written and spoken) at all ages.

School curriculum and attainment targets. In mathematics it is often essential to allow pupils to proceed more slowly at first in order to establish solid foundations. The flexibility referred to in this section is only valuable if it allows schools and teachers to adopt a pace of learning that optimises students' *ultimate* attainment—even if this involves postponing some material *from one key stage to the next.*

4. *Key Stage 1: Programme of Study*

4.1. *Year 1*

Number—number and place value.

Notes and guidance. We begin with a general observation. It would appear that the 'Notes and guidance' were originally drafted in the form of a right hand column, where each note was positioned *alongside* the content statement in the left hand column that it was meant to elaborate. The document was then edited in a way that re-located the 'Notes and guidance' to form a *subsequent* list, partly separated from that to which each note referred. Schools may therefore find it helpful to begin by deciding what each 'Note' refers to, and then relocate it alongside the relevant statutory content.

Some of the non-statutory comments in this section could have been better worded. For example:

> "Pupils practise counting (1, 2, 3, ...)"

rightly focuses attention on the mundane work that is needed both to achieve a confident grasp of the number sequence, and to overcome the confusions that arise in crossing boundaries (from 9 to 10, from 19 to 20, from 99 to 100). However,

> "practise [...] until they are fluent"

could convey the unfortunate impression that repeated counting practise will somehow *automatically* resolve any potential confusion.

Number—addition and subtraction. The extent to which teachers have been left to "join the dots" is illustrated by the treatment of "0"—both as a *placeholder* in our numeral system, and as a *number* in its own right.

(a) Zero appears from nowhere in the very first bullet point of the first section ('Number and place value') as a possible starting point for counting. There is no indication that one first has to give meaning to this intruder "0" if such a counting process is to be more than a mere ritual.
(b) The section on 'place value' makes no explicit mention of the central importance of zero as a placeholder, or of the challenges this presents to pupils.
(c) Zero then disappears from view.
(d) Zero re-appears briefly in the 'Notes and guidance' for the *present* section as a full blown *number* which can be added to, or subtracted from, other numbers.
(e) It then disappears once more—only to resurface as a number like any other in Year 2 'Number and place value', whose 'Notes and guidance' end with the curious remark that pupils will now somehow "begin to understand zero as a placeholder"—even though there has been no specified preparatory work to this end.

Number—fractions.

Notes and guidance. At a stage where small positive integers still remain mysterious, work with *fractions* needs to be handled with care. Colloquial fraction *language*—as indicated in the content listing, where 'half' is used as an adjective, or operator, to indicate "one of two equal parts" of a familiar object or quantity—is fine. But the 'Notes and guidance' appear to go further.

> "Pupils are taught half and quarter ... "

would make sense (at an appropriate stage) *provided* it is the first stage of a development that moves on later to consider 'thirds', sixths, 'fifths' etc. There is an occasional hint of such development, but too often it remains a preoccupation with 'halves' and 'doubling' that is never extended to trebling and quadrupling (an impression that is reinforced in the 'Notes and guidance' to the previous section 'Number—multiplication and division').

What seems to be missing is a coherent didactical message about what this work with fractions needs to be based on, and how it evolves: for example, how fractions *as numbers* emerge from work with integers, and from language that is immersed in simple fractions *as operators* (where a 'half' is always "half of something"). Precisely how pupils and teachers are to make this step—from thinking and talking of fractions as *operators*, to working confidently with fractions as *numbers*—is never clearly delineated.

Measurement. It is hard to assess this section in a balanced way. The list reflects what one might expect to be addressed in Year 1 in a relatively privileged school, where language is not a problem. However, one suspects that even in such a setting, the listed topics would only be *introduced* in Year 1, and would then need to be reinforced in subsequent years. In contrast, the programme of study for Year 2 here gives the impression of marching relentlessly onward—with nary a backward glance. The list of content is also disappointingly silent about pupils engaging in drawing, and copying with specified goals on accuracy, as an important part of the process of internalising the idea of measures.

Geometry—properties of shapes.

Notes and guidance. "Pupils" we are told

> "know that rectangles, triangles, cuboids and pyramids are not always similar to each other".

The grammar and wording are such that it may not be entirely clear what was intended, and one suspects that most primary teachers do not understand the mathematical notion of *similarity* sufficiently to interpret for themselves. Hence a more useful comment might have been:

> "Pupils know that a given pair of rectangles may differ both in size and shape (e.g. a long skinny rectangle, and a square), whereas another pair of rectangles may have different *sizes* yet 'look similar'; the same distinction applies to pairs of cuboids, pairs of triangles, pairs of pyramids, etc.".

4.2. *Year 2*

Number—number and place value.

Notes and guidance. 'Notes and guidance' are included in order to elaborate the succinct statutory statements of content. So it is disconcerting if the comments made seem to *obscure* rather than to clarify the listed content. In particular, the meaning of such comments as:

> "pupils [...] develop further their recognition of patterns within the number system and represent them in different ways, including spatial representations"

remains unnecessarily opaque.

What pupils (and teachers) need is a growing awareness of the key *objective structures* that have to be internalised. So, rather than leaving "recognition of patterns" unspecified, it would be more helpful to say which of these "patterns" stand out as being worth recognising in Year 2. In particular, one might expect explicit mention of

- *even* numbers, corresponding to arrangements of a set of objects into two equal rows, and
- *odd* numbers, where one can only produce two 'almost equal' rows.

It would also help to see mention of *colouring* the positive integers (and zero) on a number line using two alternate colours—which might then help to break the preoccupation with "doubling and halving" by silently suggesting that one might later use *three* colours to capture the different remainders on division by 3 (and which could then link with the suggestion in these 'Notes and guidance' that pupils "count in multiples of 3").

It would also be good to know explicitly whether the relevant "patterns" include introducing the "100 square" in Year 2.

Number—addition and subtraction. There is a serious problem with the fourth and fifth main bullet points.

In the fourth bullet point, someone may have got carried away by their knowledge of the word *commutative*. The word first becomes relevant only *for multiplication*, and then only when one begins to work with entities (such as square matrices) for which the natural binary operation **unexpectedly *fails*** to be 'commutative'. Until that point the behaviour of operations remains unproblematic, and there is no obvious reason to draw attention to this fact by inventing a special term for "the dog that didn't bark".

Moreover, in Year 2 "subtraction" is not yet a *binary operation*. Given a pair of numbers, subtraction can only be imagined 'one way round'—so *the question of whether **subtraction** is 'commutative' simply does not arise*. For children and teachers at this stage (long before negative numbers have been mooted) there is no need to mention the obvious fact that "subtraction of one number from another" depends on the order of the numbers. This bullet point should therefore be changed to read:

- "show that addition of two numbers can be carried out in either order without affecting the result."

The fifth main bullet point also gets ahead of itself. For at this stage, the "relationship" between addition and subtraction is only *partly* "inverse". The point being made would be just as effective—and more accurate—if the word "inverse" were dropped.

Notes and guidance. The final sentence of the second paragraph should be removed. One can imagine circumstances where the word "commutative" could be enlightening (e.g. for teachers of students in Years 10–12 who have just met matrix multiplication). It is harder to imagine circumstances where one would wish to draw attention to "associativity". Hence these words are completely out of place in the 'Notes and guidance' for Year 2. The claim that

> "This establishes the commutativity and associativity of addition"

would be better reworded as

> "They realise that addition of two numbers can be carried out in either order, and that addition of three or more numbers can be 'bracketed' in different ways without affecting the result."

Number—multiplication and division. Earlier public drafts of the revised curriculum included the embarrassing requirement that pupils should be taught

> "to solve mathematical formulae".

Formulae cannot be "solved"—so it is good to see that this blemish has been removed.

Unfortunately the final orders include (here and elsewhere) the unfortunate requirement for pupils to

> "calculate mathematical statements"

—which is equally impossible. This may sometimes have arisen because two incompatible requirements have been combined with a view to compression; but there is no such explanation here. Hence the second bullet point needs to be rewritten—perhaps as:

> "calculate using multiplication and division within the multiplication tables and write them ..."

The third bullet point needs to be redrafted as for 'addition and subtraction'—to remove the word "(commutative)", and to suppress the remark:

> "and division of one number by another cannot" [be done in either order]

since "division" is not yet a binary operation (that is, it is only defined for certain ordered pairs).

Notes and guidance. The second paragraph would be much clearer if it added the words "into minutes" to read:

> "[Pupils] connect ... the 5 multiplication table with the divisions *into minutes* on the clock face."

In the third paragraph, the final sentence needs to be redrafted as for "addition and subtraction" to remove all reference to the word "commutativity", and preferably to drop the word "inverse". It is also completely unclear what is meant by the expression "multiplicative reasoning" at this level. The following might be clearer and more helpful:

> "They relate multiplication and division, and begin to see the two statements '$4 \times 5 = 20$' and '$20 \div 5 = 4$' as two aspects of the same numerical fact."

Number—fractions. It is good to see, despite the typographical challenge, that fractions have all been typeset 'vertically'.

It is less encouraging to find that the statutory curriculum and the 'Notes and guidance' consistently reinforce the impression that the approach to the teaching of fractions has been fudged—as though no one will notice the circularity of:

> "Pupils use fractions as 'fractions of' discrete and continuous quantities".

There is no indication as to how teachers and pupils are supposed to make the crucial step from 'fractions as operators' to fractions as *numbers* with their own arithmetic. The consequences of this very English fudge are extensively documented in such studies as ICCAMS and TIMSS.

Notes and guidance. The opening sentence should perhaps read:

> "Pupils use fractions as operators—that is, as 'fractions of' discrete and continuous quantities. They solve problems involving shapes, objects and quantities."

The second paragraph combines two assertions that need to be separated. The opening sentence is fine. But the first half of the next sentence may be wishful thinking: there is much to be said for

> [counting] "in fractions up to 10, starting from any number";

but it is hard to see how such recitation will "reinforce the concept of fractions as numbers". The second half of the final sentence makes a useful point—if one that may be somewhat premature in Year 2, where fractions remain "parts"—and hence less than a whole. It is true that the idea that one can go beyond a whole is implicit in the process of counting in fractional steps; but it would be much clearer if it was stated as a *separate* sentence:

> "Pupils realise that when adding fractions (whether as numbers, or as parts of a repeated whole), the result can be greater than 1."

Measurement. The continued emphasis on measurement is welcome. However, it has already been suggested that the total listing for Year 1 and 2 may prove to be over-ambitious. And there should be much more emphasis on drawing, with specified goals on accuracy (such as "drawing line segments accurate to the nearest centimetre") and on estimating ("estimating the length of given line segments") in order to slowly establish an internal mental yardstick.

The features that are common to all measures—and that make it possible to calculate numerically with measures—need to be emphasised more clearly, and the connections with other parts of the programme of study need to be underlined. That is, the "connections" which were trumpeted in the initial 'Aims' (see Section 3 above) need to be brought out in the listing of content.

There are two serious flaws in the listed content.

The 'Notes and guidance' to the Year 1 programme of study declared that

> "mass and weight are used interchangeably".

This may appear balanced and broad-minded. However, the truth is that, whereas pupils in Years 1 and 2 can physically experience, and hence appreciate "weight", it is rather hard to imagine ways in which they might make sense of the much more subtle notion of "mass" (which even Newton found elusive). Yet already in Year 2 the very first bullet point reverts exclusively to the language of "mass". The extent of this confusion becomes more intriguing when we read in a later bullet point that pupils are to measure

> "mass [. . .] using scales".

One suspects that most equipment which could be described as "scales" is actually designed to measure *weight* rather than mass. (Some may claim that the word "scales" here includes the idea of a balance. But a balance only allows one to *compare* masses—in fact by comparing their respective weights! A balance does not directly "measure" anything at all.)

Much more disturbing is the way "temperature" is presented as if it were a genuine measure. It isn't. This confusion is common in textbooks and assessments, but one would prefer not to see it replicated in a national curriculum.

Temperature seems to have a numerical aspect, so it is natural to think it should be part of school mathematics. Yet appearances can be deceptive.

In reality it is not entirely clear how temperature should be handled at school level. The mathematics of 'temperature' seems to fit somewhere between the mathematics of bus numbers (where the numbers are arbitrary labels that should never be added or doubled), and the mathematics of *time* (which has its own difficulties, but which is closer to a being a genuine measure than is temperature). One cannot meaningfully double a bus number, or a time, **or a temperature**.

Temperature is quite different from true 'measures'—such as *length or weight.*

True measures are based on a choice of unit that can be easily *replicated.* Two '1m rulers' can be laid end to end to make 2m; three identical 1kg weights can be placed together in a scale pan to make 3kg. In both cases the basic unit can also be naïvely subdivided into equal parts to obtain "half a kilogram", or "one hundredth of a metre".

True measures also have a natural *zero.* And this together with replication and subdivision of the unit allows us to multiply measures by numbers (e.g. to get twice the unit, or 'three halves' of the unit).

This means that true measures *give rise to numbers*—and the resulting numbers can be subjected to calculation using familiar arithmetic: that is, true measures form a 1-dimensional numerical scale.

Much of this breaks down for temperature. The relevant notion of 'unit' (what is meant by 1°) is beyond the comprehension of a Year 2 pupil. There is no naïve way of replicating, or of subdividing the unit. And there is no zero. Hence there is no easy way to make sense of 'calculations' in the context of temperature: there is no way to add temperatures, or to make sense of the words "twice as hot".

The same is partly true of *time.* But 'time' is more amenable to calculation than is temperature. There is again no 'zero'. But there is a perfectly accessible—if philosophically problematic—uniform scale which allows one to choose an origin for the purpose of addition and subtraction, or difference, and which allows us to make sense of expressions such as "twice as long".

All of which highlights the absence of, and the need for, a more careful "didactics of measures".

Notes and guidance. The second paragraph of the subsequent 'Notes and guidance' illustrates once more the fixation on "half as" and "twice as"—ideas which could be excellent beginnings as long as they routinely lead to considering "one third as" and "3 times as", etc..

We are told that Year 2 pupils

> "become fluent in telling the time on analogue clocks [to the nearest five minutes] *and recording it*".

It would help to know whether they are still expected to draw pictures of the clock face, or at what stage they are expected to access a more compact way of recording.

Geometry—properties of shapes. The four bullet points are well-intentioned, but curiously inadequate.

If we refer back to Year 1, we see that the only named 2D figures that have been introduced are

> "rectangles, circles and triangles".

So one would expect the Year 2 list to indicate explicitly how these 'small beginnings' are to be further *explored and extended*. In fact all we see here is a vague reference to "2D shapes", with some elaboration in the 'Notes and guidance'. This gives the (false) impression that the list of additional shapes is in some sense immaterial—whereas it is important and needs to be clearly specified.

The third bullet point also starts well, but falls short. It is perhaps natural for pupils to focus in the first instance on the 'surface' of a 3D shape: e.g. to see the square faces of a cube, or the circular end of a cylinder. But mathematics often depends on learning to see things that are not immediately apparent, and the curriculum should explicitly highlight this fact, so that teachers know they have to look beneath the surface. The eventual mathematical analysis of 3D solids depends rather little on what one can see on the 'surface', and is much more concerned with *2D cross-sections*. Hence it is important to clarify when teachers should begin to cultivate this kind of deeper perception in their pupils.

It also remains a mystery as to why we persist in producing curricula that fail to incorporate clear requirements with regard to 'drawing, making and exploring'. The 'Notes and guidance' mention "drawing lines", but this is clearly a rather limited substitute for "drawing, making and exploring", and underlines our lack of a clear didactical sense of how "mental objects"— especially in mathematics—derive their eventual robustness from engaging the hand, eye and brain in suitable prior activities.

Geometry—position and direction. It is unclear why 'Geometry' is subdivided into two sections—neither of which has a coherent structure. One has to hope that schools will amalgamate the two headings and look for ways of giving the combined theme some clear 'sense of direction'.

5. *Key Stage* 2

5.1. *Lower Key Stage* 2—*Year* 3

The introductory paragraphs which precede the list of content for lower Key Stage 2 are somewhat mixed. The opening sentence is not unreasonable—though it would have been good to see some reference to measures and to engaging with word problems rather than just to "the four rules".

The second paragraph refers non-specifically to solving

> "a range of problems",

and emphasises

> "including with simple fractions and decimal place value".

The recent ICCAMS study demonstrated how limited is the ability of **Key Stage 3** pupils in England when faced with the simplest problems involving fractions and decimals. So while there is scope in Years 3 and 4 for using the language of fractions and fractional parts, the

> "ability to solve problems with fractions"

may prove rather limited at this stage.

In contrast, one would expect decimals, and measures using decimals, to play an increasingly important role in throughout Key Stage 2. So it is remarkable that, having declared their importance in this introduction, the list of content repeatedly misses the opportunity to clarify where decimals should be addressed, and says nothing to alert teachers to places where decimals, and decimal place value, need explicit and careful teaching. In particular there is no hint that decimals play a key role in linking number and measures.

The statement in the introduction that pupils should "use measuring instruments" is then scarcely developed in the list of content.

The requirement for pupils to "memorise" multiplication tables is welcome. "Memorise" means pupils can be expected to have "instant recall", rather than the much weaker expectation that they should be able to work out those products which remain unlearned. However, many pupils will not manage to do this by the end of Year 4 as stated here.

The final paragraph is deeply disappointing. The requirement that

> "pupils should read and spell mathematical vocabulary correctly"

does not begin to recognise the profound interdependence of language and mathematics at this level. Not only is the logic of mathematics understood and expressed through the grammar of spoken English, but actions (such as "add" and "makes") are verbs, calculations (such as "$13 + 9 = ??$") have the structure of sentences, mathematical properties are captured through the use of adjectives, and the ability to solve problems requires pupils to engage with the process of extracting meaning from descriptions given in words—that is, through 'comprehension'. These matters all warrant a much more detailed emphasis than does 'spelling'.

Number—number and place value. The list of content appears reasonable—but the lack of a clear underlying didactical thrust makes it hard to be sure what else is needed or intended. For example, there is no mention of *decimals*—which should surely appear in a section headed 'place value' if anywhere. This might indicate that decimals are not expected before Year 4. Yet the section on 'Measurement' gives the impression that decimal measures are bound to arise; and the 'Notes and guidance' to the section on 'fractions' include the statement that

> "pupils connect tenths to place value, decimal measures and division by 10".

Notes and guidance. We are told that

> "pupils now use multiples of 2, 3, 4, 5, 8, 10, 50 and 100".

The general thrust of this requirement is fine—though some of these (e.g. 8) may well be premature. Hence our comments here (and for Year 4 below) focus more on the *relative* order in which multiples are addressed as part of the preparation for eventual mastery of the relevant multiplication table, rather than on insisting which multiples belong in Year 3 and which in Year 4. For example, one suspects that 11s may be rather easier to handle than 8s, so 'counting in 11s' may fit in before 'counting in 8s' (with 'counting in 12s' delayed until later).

Within the possible list 2, 3, 4, 5, 10, 11, 50, 100 of simpler multiples, 2, 5, and 10 stand out as being especially important given our 'base 10' numeral system, because $2 \times 5 = 10$.

In the same spirit, 2 and 50 are important because $2 \times 50 = 100$. And for the same reason one would expect to include 20 since $5 \times 20 = 100$. But these two observations overlook the more natural extension of $2 \times 5 = 10$ to three figures—namely $4 \times 25 = 100$. The common neglect of "multiples of 25" helps to explain why very few pupils appreciate that, just as 2s and 5s go together when dealing with multiplication, so in the same spirit "4 loves 25" (a fact which extends later to $8 \times 125 = 1000$, and links to recognising the decimal for $\frac{1}{4}$ as 0.25, and that for $\frac{1}{8}$ as 0.125). So 25s should be included sooner rather than later—though perhaps in Year 4 rather than Year 3.

"Partitioning related to place value" should be explicitly mentioned as statutory content.

Notes and guidance. The 'Notes and guidance' should highlight the fundamental nature of the decomposition

$$146 = 100 + 40 + 6$$

into Hundreds, Tens, and Units, rather than lumping it together with the arbitrary, or incidental, partition $146 = 130 + 16$.

One assumes that the word "and" in the curious 'equation':

$$\text{“}146 = 100 + 40 \text{ and } 6\text{”}$$

should have been written as a "+".

Number—addition and subtraction. The expression "columnar addition" in the second bullet point (and elsewhere) is non-standard. It would be better to refer consistently to "*column* addition".

The fourth bullet point again misses the opportunity to emphasise *word problems*. The primacy of English means that mathematics and mathematical thinking have to be accessed through language—especially in primary school. We need to learn from countries that prepare their pupils for secondary school much better than we do. Such countries build their primary mathematics around a central core of word problems. They use these problems to strengthen pupils' grasp of language, and the meanings they attach to mathematical terms and ideas. And they use such problems systematically to teach pupils how to think, how to analyse given information, and how to solve serious problems *before* general-purpose techniques such as algebra are available later in Key Stage 3.

Number—multiplication and division. There is an apparent lack of joined up thinking between 'Number and place value' and 'Multiplication and division'. The activities of counting first in 2s, and 5s, and 10s, and then in 4s, and 5s, and 10s, and 20s, and 25s, and 50s, and 100s, should lead naturally to an emphasis on the related products

$$2 \times 5 = 10, \text{ and } 2 \times 50 = 4 \times 25 = 5 \times 20 = 10 \times 10 = 100.$$

The second bullet point accidentally repeats the howler of requiring pupils to "calculate mathematical statements". This time it may be partly because two ill-matched requirements have been combined. The bullet point should read:

> "write mathematical statements for multiplication and division and calculate using the multiplication tables that they know, including ...".

The final part also tries to make the single word "methods" work overtime, and would be much clearer if the word were repeated:

> "using mental methods and progressing to formal written methods".

The second bullet point and the associated 'Notes and guidance' require a degree of interpretation. The statutory content refers to "multiplication and division", but then only mentions

> "two-digit numbers *times* one-digit numbers":

this leaves rather little scope for division at this stage. So the final remark

> "using mental methods and progressing to formal written methods"

would seem to indicate that work in Year 3 should aim to lay foundations in short multiplication, which may then be further developed in Year 4 ("progressing to"). Hence the final comment in the third paragraph in the 'Notes and guidance' would appear to be one year too early, and might be better expressed as:

> "starting with ... and progressing **in Year 4** to the formal written methods of short multiplication and division."

Having specified purely numerical exercises in the second bullet point, the third bullet point needs to make it clear that the problems to be solved are not just 'more of the same'. This would be relatively straightforward if we had a consensus about the constituent stages in the didactics of solving problems which allowed one to flag this distinction by referring to "*word* problems", or to "multi-step problems". The bullet point could then be clarified as

> "solve *word* problems and multi-step problems involving multiplication and division, including missing number problems, positive integer scaling problems, and systematic counting problems ...".

One can imagine exercises involving scaling, but it is unclear what kind of problems will be construed from the phrase

> "including positive integer scaling problems".

(The best I could come up with was to ask about the combined effect of starting with a given figure and enlarging in turn using scale factors of 2 and then 3, or using scale factors of 3 and then $\frac{1}{2}$.)

The final requirement to include

> "correspondence problems in which n objects are connected to m objects"

should perhaps have been expressed concretely (as in the ensuing 'Notes and guidance'):

> "and systematic counting problems which enumerate all possible links between one or two small groups (e.g. where 4 men all shake hands with each other, or where 4 men all shake hands with each of 3 women)".

Notes and guidance. The reminder that repeated doubling links the 2, 4 and 8 multiplication table is fine. But one would also like to see later mention that doubling links the 5 and 10 multiplication tables and the 3 and 6 multiplication tables, and that *tripling* links the 2 and 6, and the 3 and 9 multiplication tables.

The examples given in the second paragraph miss the opportunity to emphasise *products that make* 10 (or 100). If pupils are to become fluent with the decimal number system, they need to look for products like 2×5, or 4×25, or 5×20, and to decompose

$$4 \times 12 \times 5 = 4 \times 6 \times (2 \times 5) = 4 \times 6 \times 10.$$

Numbers—fractions. The listed content is all very nice—but it may prove premature for many pupils. The first, second, and fourth bullet points appear to focus on "fractions as parts of", while the third and fifth bullet points relate to "fractions as numbers". But there is no apparent recognition of the tension between these two perspectives, how they are related, which is ultimately the more important, and how one can eventually be subsumed within the other.

Notes and guidance. The expected link to decimals will require preparatory work under other headings—even though this is not explicitly mentioned elsewhere.

The assertion in the second paragraph that "they should go beyond the $[0, 1]$ interval" matches the comment made in the 'Notes and guidance' for Year 2, but does not sit easily with the spirit of the statutory requirements for Year 3, which convey a clear impression of staying "within one whole".

The statement that fractions should be "related to measure" is welcome.

Measurement. One might expect some explicit mention of the need for work on 'reading scales'.

It is good to see perimeter introduced on its own (rather than being linked to area, which is a much more difficult notion).

There needs to be some explicit mention of pupils beginning to grapple with simple decimal measures.

Notes and guidance. The second paragraph looks like a 'note' that never got refined: scaling does link to multiplication, as does changing units (e.g. from metres to centimetres, as in the first paragraph); but comparison of measures has no obvious connection with scaling or multiplication.

Geometry—properties of shapes. The subheading "properties of shapes" is now redundant—since the separate headings at Key Stage 1 have here been effectively combined into a single 'Geometry' section, which incorporates both 'properties of shapes' and 'position and direction'.

The listing lacks detail: "draw 2D shapes" fails to specify which 2D shapes play the most important role, or which geometrical perceptions and skills need to be developed.

Notes and guidance. The final non-statutory comment

> "pupils connect decimals and rounding to drawing and measuring ... in centimetres"

suggests that the statutory listing should specify

> "draw 2D shapes, including straight lines to be drawn (to some specified accuracy)".

It also suggests that something similar needs to be specified for angles (in Year 4 or Year 5).

The welcome (non-statutory) exhortation to

> "connect decimals and rounding to drawing and measuring"

draws attention to one of the many links which teachers will need to infer from the otherwise rather stark listing of content.

Statistics. It would be good to see an emphasis on keeping the width of the 'bars' in a bar chart fixed, and a 'Note and guidance' comment referring to the fact that, if the bars all have the same width, then it is not only the *height* of each bar that represents the size of each group, but also the *area.*

5.2. *Lower Key Stage 2—Year 4*

The programme specified for Year 4 (and beyond) is poorly chosen, poorly constructed, and poorly expressed—and to such an extent that constructive line-by-line comment becomes increasingly awkward. One is left with the impression that an unsatisfactory process resulted in time pressure and too many unseen hands intervening to 'improve' the draft, with the result that no one took ultimate responsibility for the published programme of study. This makes it difficult to identify and to highlight a few main points that need serious attention—for, in the absence of an underlying didactical architecture, the conception of what constitutes primary mathematics easily degenerates into a disorganised list.

Number—number and place value. In the first bullet point, it is good to see 25 put in an appearance—though for reasons explained in Year 3, it should perhaps be given a higher priority than 'counting in 7s or 9s. And the final phase of this sequence of stages for counting in steps should probably include 'counting in 12s' (either in Year 4, or perhaps in Year 5 if mastering multiplication tables proves to be more realistic in Years 5 and 6).

The third bullet point is very hard to understand. We have already noted that the lack of a didactical framework for introducing and establishing an understanding of zero has given rise to curriculum requirements that are conjured out of nowhere, and that are often needlessly premature. This neglect of an underlying framework is even more serious for *negative numbers*: so one is left with the impression that all pupils are expected to do is to recite "minus 1, minus 2, etc." without any grasp of what such words might represent. (For what it is worth, the Singapore primary curriculum† includes an impressive amount of work on fractions and algebra—*but does not include a single mention of the word "negative".*)

Just as it is natural to use the simple language of fractional parts ("half a pint", "half an hour") long before treating fractions within the mathematics curriculum, so pupils can expect to meet negative numbers incidentally in certain contexts long before they are ready to address the mathematics of negative numbers. This item seems to stem from a belief that ritual incantation in the absence of meaning can somehow magically lead to meaning and mastery. In fact, **the arithmetic of negative numbers is not needed before Year 7 and is best left to Key Stage 3**. (Unfortunately, the didactical framework of negative numbers is completely invisible at Key Stage 3, where they receive not a single mention.)

The sixth bullet point remains opaque. If the intended content matters, then it should perhaps be more clearly expressed.

The final bullet point is again very poorly worded. The idea that children should learn to read Roman numerals is relatively harmless—except that there are hundreds of other things that may claim an equal right to being mentioned, so their inclusion contradicts the declared aim of focusing on 'core' content. Like many other topics, Roman numerals might warrant a mention in the mathematics classroom—and may even feature in the occasional puzzle; but they have no role to play in school mathematics (for example, they receive no mention in the Singapore primary mathematics curriculum). And it is unfortunate that the bullet point has been so worded as to suggest *wrongly* that the Roman numeral system

> "changed over time to include the concept of zero and place value".

It did no such thing. A more accurate formulation might be:

> "read Roman numerals to 100 (from I to C), and contrast these with our own Hindu-Arabic numeral system based on place value and a built-in concept of zero".

Notes and guidance. The first paragraph looks like an attempt to combine two alien worlds—on the one hand the world of 'fluency', 'repetition' and 'practise' and on the other the world of understanding and meaning. The result can be confusing, for the attempt to inject a modicum of understanding and meaning often relies on

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†Ministry of Education Singapore. Mathematics Syllabus Primary, 2007. `http://www.moe.gov.sg/education/syllabuses/sciences/files/maths-primary-2007.pdf`.

repeated words and phrases—such as “variety (or range) of” and “representations”—whose meaning is unclear. Despite the confusion, the aim of injecting meaning is welcome, since achieving an effective grasp of ‘order’ and ‘place value’ does require pupils to engage with the different ways in which number arises (which, at this level, mostly means through counting, measures and pure number).

The second paragraph refers to “decimal numbers” as though unaware that the statutory requirement refers only to whole numbers, and makes no reference to the challenge of extending pupils’ world of number to include decimals.

The final paragraph fails to clarify that Roman numerals have nothing to do with mathematics precisely because the Romans failed to incorporate either zero or place value into their numeral system. (Intriguingly, despite their many strengths, the Romans contributed almost nothing to mathematics.)

Number—addition and subtraction. In Year 4 one expects that prior work with whole numbers and measures will begin to feed in to work with decimals and fractions; yet the heading ‘addition and subtraction’ continues to focus exclusively on whole numbers. And the three bullet points convey the impression that the only development in Year 4 is to work with slightly larger numbers. This is a serious mistake. One can perhaps understand having a separate heading for fractions, but the present section really needs to embrace the progression from addition and subtraction of whole numbers to confront the challenges of working with *decimals*.

As competence and confidence increase, the range of problems which come within pupils’ reach broadens. So as well as classical word problems (which are much more general than the “two-step problems in context” referred to here) one might like to see mention of ‘missing digit’ addition and subtraction problems.

Notes and guidance. One would welcome ‘Notes and guidance’ that sought to enrich pupils’ experience. But one’s heart sinks when the only additional observation is that

> “Pupils continue to practise ... to aid fluency”.

Number—multiplication and division. The first bullet point

> “pupils ... recall ... multiplication tables”

should be clear, since it was stated in the preamble to the Key Stage 2 programme that pupils are expected to “memorise” their tables. Hence “recall” means *instant* recall. However, inserting this requirement prematurely in Year 4 invites well-meaning attempts to soften the blow—as one sees in the ‘Notes and guidance’ which include the curiously woolly construction

> “pupils continue to practise recalling”.

The second, third and fourth bullet points provide an excellent example of how a simple list can effectively convey, *or fail to convey*, important didactical connections. The second and fourth bullet points are closely connected, since laying out “307×8” in column format is the most obvious place where a Year 4 pupil will need to “multiply by 0”. In contrast, the third bullet point is completely unrelated to the second and the fourth—and inserting it between them makes it much harder for teachers to notice the connection. Instead of gratuitously splitting them, the related bullet points should be juxtaposed and worded to enhance the impact:

- "use place value, known and derived facts to multiply and divide mentally, including: multiplying and dividing by 0 and by 1 (e.g. as required when implementing the short multiplication of 307×8); dividing by 1;
- multiplying together three numbers multiply two-digit and three-digit numbers by a one-digit number using column layout for short multiplication".

The third bullet point imposes the C-word (commutative) on what "pupils should be taught". This is wrong (the Singapore primary curriculum knows this and the word does not appear). Not surprisingly the requirement ignores the fact that the more important property when exploiting factor pairs is the *associative law*. This is the first appearance of the expression "factor pairs"—and even of the word "factor". So perhaps the requirement would be better formulated as:

> "factorise numbers and combine factorisation and rearrangement to simplify written and mental calculation (e.g. $12 \times 75 = 3 \times (4 \times 25) \times 3 = 900$); recognise and use factors and factor pairs"

The final bullet point with its reference (repeated from Year 3) to

> "correspondence problems such as n objects are connected to m objects"

seems to have missed out on a final editing. The expression "correspondence problems" seems out of place here—as does the use of algebraic symbols. A clearer version might read:

> "including integer scaling problems, counting dots in rectangular arrays, counting ordered pairs (e.g. the number of possible meals with three main courses and four possible second courses, or the number of possible outcomes when two or three dice are rolled)"

Notes and guidance. The emphasis on

> "pupils continue to practise ... to aid fluency"

would be easier to welcome if these words were not so clearly overused. Practice is more important than we may have recognised, but it needs to take place within a richer didactical framework than is displayed here.

The second paragraph makes a common, but inexcusable, jump in claiming

> "pupils ... derive facts (for example $600 \div 3 = 200$ can be derived from $2 \times 3 = 6$)".

The process of helping pupils to see a connection between these two apparently unrelated statements is what a programme of study should map out, and requires a clear didactical architecture which this particular version lacks.

The third paragraph continues in the same heartless vein: it declares that

> "Pupils practise to become fluent in the formal written methods of short multiplication and short division."

This is a non-statutory note on the preceding statutory list of content. Yet it fails to notice that the preceding statutory list includes nothing that could possibly suggest "short division".

The fourth paragraph elaborates on the kind of statements pupils write. But in choosing to present the uninteresting fact

> "$(2 \times 3) \times 4 = 2 \times (3 \times 4)$",

the guidance misses an opportunity to show the advantages of re-bracketing by presenting an example that exploits the decimal structure of our numeral system, e.g.

$$(3 \times 4) \times 25 = 3 \times (4 \times 25) = 300.$$

There is also no explicit statement that the words "commutative", "associative" and "distributive" are directed towards teachers rather than pupils. Nor is there any subsequent indication that this potential reference to the *structure* of arithmetic is meant to be developed in Years 5 and 6.

The final paragraph suggests the inclusion of

> "questions such as ... three cakes shared equally between 10 children"

but does so without explaining that this links the present statutory section (which is restricted to whole numbers) and the next section (on fractions or decimals).

Number—fractions (including decimals). As we move through the Year 4 programme of study it becomes increasingly difficult to infer the intended thrust.

The first three bullet points make a kind of sense (though they may well prove premature).

The fourth bullet point is likely to evoke a yawn. In some sense, one understands why Year 4 pupils are restricted to

> "adding and subtracting fractions *with the same denominator*".

But one is then obliged to ask:

> Why was the same activity already begun prematurely back in Year 3?

The question is even more pressing when one looks ahead to Year 5 and sees how slow subsequent progress continues to be.

This is a common English disease: content is introduced too early, and then repeated in 'baby' form year after year, with no significant progression. In contrast, effective education systems think carefully how to prepare the ground well, so that when the full-blooded topic is finally introduced, significant progress can be made relatively quickly. For example, Singapore focuses first on relevant preparatory work with whole numbers, and then concentrates on comparing—and later on calculating with—fractions having denominators ≤ 12 where all the essential ideas can be grasped in a relatively familiar and restricted setting, before extending fraction work to the general case.

I have noted the long silence about extending the numeral system and its arithmetic to decimals. So it is remarkable that the next six bullet points suddenly impose jaw-dropping requirements on what "pupils shall [suddenly] be taught". Despite its extent, the list is half-baked. Pupils are expected to

> "divide a one- or two-digit number by 10 or 100, identifying the value of the digits in the answer as ones, tenths and hundredths",

but there is not a word anywhere about addition or subtraction of decimals. No one would know that the recent ICCAMS study of 14–15 year olds found that very few of those completing Key Stage 3 (around 16%) responded correctly to the question:

> "We write six tenths as a decimal as 0.6. How do we write eleven tenths as a decimal?"

Any future programme of study—and any intelligent interpretation of the present version—will need to address these matters more effectively.

Notes and guidance. The second paragraph is unclear. One can see ways to use the number line to link fractions and decimals; but the inclusion of "and measures" seems out of place.

The third paragraph is a bald assertion:

> "pupils understand ...".

Given the absence of any didactical structure to help develop the required links, it may not be this easy—for the crucial link between fractions, decimals, and division is never clarified.

It is hard to be sure what "connections" the fourth paragraph intends pupils to make

> "between fractions of a length, of a shape and as a representation of one whole or set of quantities."

And the second sentence seems to take an unhelpfully procedural view. Before

> "pupils [can] use factors and multiples to recognise equivalent fractions",

they first need to understand what it is (other than formal cancellation) that makes $\frac{10}{15}$ and $\frac{14}{21}$ "equivalent". There is almost no sign of the necessary emphasis on the crucial defining property of *unit fractions*.

The fifth paragraph brings us down to earth with a thud: "pupils" we are told yet again

> "continue to practise ... to become fluent".

This suggests they may well be subjected to a more rigorous regime than those pupils who are currently in Year 4; but it is hard to believe that such a regime will leave them much wiser.

The sixth paragraph is likely to cause some confusion. The statement that

> "pupils are taught ..."

has a Dickensian, Gradgrind ring to it—for there is precious little indication as to how the poor souls are to make sense of what they may be told. The note also misuses the word "proportions"— confusing it with the word "ratio". And it also uses the word "are" when it would be more accurate to say "can sometimes be" (when a ratio is interpreted numerically, it gives rise to a decimal or a fraction—but decimals and fractions also arise in other ways).

The final line of the seventh paragraph has a missing word:

> "division of a whole number by 10 and later by 100".

It is good to stress this connection between decimals and the result of dividing an integer by some power of 10. But this underlines the need for some guidance as to how the idea might be understood at this level (other than as a rule)—which brings us back once more to the missing didactical architecture, and the importance of the fundamental property of unit fractions (namely that

$$\frac{1}{2}+\frac{1}{2}=1, \quad \frac{1}{3}+\frac{1}{3}+\frac{1}{3}=1, \quad \frac{1}{4}+\frac{1}{4}+\frac{1}{4}+\frac{1}{4}=1,$$

etc.).

The eighth paragraph is incoherent and needs to be rephrased. It should presumably say:

> "They practise counting in constant steps, both forwards and backwards, where the step may be a fraction or a decimal."

At the right time and in the right place, and provided it is used sparingly, this could be an excellent exercise. But the apparently endless repetition of this 'counting in steps' idea simply draws attention to the lack of any *other* didactical support. It remains unclear (at least to this reader) how this ritual on its own will deliver confidence in using (as opposed to merely *naming*) numbers.

The final paragraph needs to be reconsidered. We know that in moving from comparing integers to comparing decimals, pupils are often left trusting the old strategy of thinking that "longer (more digits) is larger"—ignoring the position of the decimal point. In trying to guarantee short-term pupil success, teachers are here advised to order

> "decimal amounts and quantities that are expressed to *the same number of decimal places*".

This seems designed to reinforce the well-known misconception. While there may be a good reason to work first with numbers having at most one decimal place, and later with numbers having at most two decimal places, one suspects that it is important to compare 137, 13.7. and 1.37, or 3, 2.4, and 2.35.

It is unclear why there is still no move towards addition and subtraction of decimals.

Measurement. The whole character of the document is summarised by the fact that pupils are taught

> "to convert between different units"

but we are not told exactly which units. (The two examples given are also presented in a curiously illiterate *singular* form "kilometre to metre; hour to minute".)

There is an eerie silence about practical work relating to volume and capacity, and activities designed to establish an internal yardstick that would allow pupils to estimate everyday weights with tolerable accuracy. And there is no mention of the active process of learning to read scales, and to interpolate between marks, which might help to give meaning to the measures that are being formally "converted".

The authority of a national document is undermined if it has clearly not been proof-read, uses key words incorrectly, or includes avoidable howlers.

The second and third bullet points go out of their way to use the slightly unusual word "rectilinear". It should be clear that it means "in a straight line". Hence a "rectilinear figure" is "a shape contained by straight lines"—in other words, a polygon (see, for example, the *Shorter Oxford Dictionary*). But that is not how the word is used here.

Modern sloppiness has given rise to various misuses which do not realise that the prefix "recti" simply means "straight", and which instead assume that "recti" has something to do with "right angles". However, one would prefer a *national* document (especially one intended to raise educational aspirations) to avoid such sloppiness. One would also prefer it to stick to language that will be familiar to primary teachers.

The clue as to the apparent nature of the error lies partly in the unfortunate parenthetical addendum "(including squares)"—since no one would ever write "polygons (including squares)". There is also a hint in the third bullet point where we are told that

> "pupils should be taught to find the area of *rectilinear* shapes by counting squares".

This suggests strongly that the shapes referred to are imagined as having their edges drawn along the grid lines of a square grid. That is, the word "rectilinear" is (wrongly) assumed to be a shorthand for

> "a compound shape made from rectangles, or from unit squares, with all sides running in just two directions (like a 'staircase', or a Swiss cross)."

No other 2D shapes are mentioned; and though the perimeters of these compound rectangular shapes are "measured and calculated", there is no other mention of drawing or measuring—whether of line segments or of angles, or of circles—and there is no work at all in 3D. These compound rectangular figures constitute an important family; but they are also rather special and somewhat unruly. And although they are tailor-made for measuring and calculating perimeters and areas, one may need to simplify by restricting to rectangles in order to notice how scaling affects area and perimeter. Moreover, the significance of this family arises from the way one can use a rectangle to derive the formula for the area of a triangle, the fact that one can cut other polygons into triangles, and the trick of using compound rectangles to approximate other polygons, circles, etc. Yet here there is no hint of preparatory work with these other shapes. That is, the lack of a clear didactics of shape and measures has obscured the important steps that lead from triangles and quadrilaterals, to polygons and circles, and then back to analysing more complicated shapes in terms of rectangles and triangles.

The result is a list of content that is restricted to specifying isolated 'methods', which are then not used to do anything interesting.

Notes and guidance. The claim that

> "perimeter can be expressed algebraically as $2(a + b)$"

is most unhelpful. First it should refer explicitly to "the perimeter *of a rectangle*". Second, it is pretentious to imagine that anything is gained by lumbering Year 4 teachers with such symbolical pseudo-science—which would be far better expressed as:

> "the perimeter of a rectangle includes two equal "lengths" in one direction, and two equal "widths" in another direction, and so can be calculated as "$2 \times$ ('length + width')."

And while it is good to see algebraic symbols appearing in italic, this is not done consistently (even within this single paragraph).

The note

> "they relate area to arrays and multiplication"

tries to make a potentially valuable connection; but it needs to be refined.

- It confounds the concept of "area" with the numerical measure that emerges.
- It fails to notice the distinction between counting dots in a 2 by 3 array (which has 2×3 dots) and calculating area (an array of $2 \times 3 = 6$ dots is likely to be perceived as enclosing *just two* unit squares).
- And it fails to notice the missed opportunity to include

  > "counting the number of dots in a rectangular array"

  in the earlier section 'Number—multiplication and division'.

Geometry. The programme appears to forget that there was only one 'Geometry' section in Year 3. Combining the two sections permanently would enhance coherence, and would avoid highlighting the desultory-looking 'Notes and guidance' for 'position and direction' in Year 4 and Year 5. I shall therefore henceforth comment on 'Geometry' as a single section.

The listed themes mostly deserve to be addressed—but the essential details are missing. And there is no clear underlying framework that might help the teacher to recognise what is missing, or to understand why these missing items matter, or how the listed items should be interpreted. In particular, within the programmes for Years 4, 5, and 6 there needs to be a greater emphasis:

- on reinforcing the language of absolute and relative position;
- on the arithmetic of angles (e.g. using problems relating to the hands of a clock)—including that angles at a point add to 360°, that angles on a straight line add to 180°, and that a right angle is 90°;
- on those "geometrical properties" that play a central role (the content listing is curiously silent about these—yet the 'Notes and guidance' assume acquaintance
  (a) with isosceles and equilateral triangles—but not right angled triangles;
  (b) with parallels and perpendiculars;
  (c) with parallelograms, rhombuses, and trapezia);
- on material relating to circles, semicircles, parts of circles, compass directions, etc.;
- on drawing and learning to accurately copy 2D configurations—from simple polygons to compound figures (with the need to measure and replicate line segments, and to copy angles—even if this is not done by measuring until Year 5);
- on making and talking about simple 3D figures.

Notes and guidance. The lack of essential detail in the statutory listing of content passes the buck to the 'Notes and guidance', which then have to compensate by indicating what has been inexplicably omitted from the statutory list. This is unsatisfactory and needs to be corrected.

The second paragraph on 'properties of shapes' introduces what looks like an unnecessary, and possibly unhelpful, distinction. The basic 2D rectilinear figure is called a *polygon*; and within the family of all polygons with a fixed number of vertices, one singles out those that are *regular*. If one is then given a polygon, one may ask whether it is "regular" or not. But it is not helpful to apply standard linguistic constructions to invent a new category of "irregular" polygons. (Mathematical convention occasionally invents such a new category. One example is the language used for *rational* and *irrational* numbers, but this reflects the fact that numbers that fail to be rational were seen to be both surprising and interesting. Hence this is a relatively rare instance.)

There is a minor *faux pas* in the Notes on 'position and direction'. We talk about the 'coordinates' of the point $(2,5)$; and the coordinates are given as an 'ordered pair'—with the "2" as the first coordinate, and the "5" as the second coordinate. But we do **not** call $(2,5)$ a "*pair* of coordinates".

Statistics. This section seems relatively harmless, but lacks a sense of priority and direction. Bar charts and pictograms may prepare for later work, but the most pressing everyday sources of data probably involve 'tables'—extracting information from tables (e.g. timetables), compiling information in effective tabular form, and discussing and making sense of information presented in a table. Yet all this seems to be hidden in the one word "tables", which therefore seems likely to receive relatively little attention.

It is also unclear—despite the mention of "time graphs" and "other graphs"—how the connection is to be forged between this initial work on representing data and later work with coordinates and graphs.

5.3. *Upper Key Stage* 2—*Year* 5

The preamble is difficult to swallow. It begins by asserting—with an apparently straight face—that

> "the principal focus [in Years 5 and 6] is to extend [pupils'] understanding . . . to include larger integers."

The assertion has the virtue of simplicity. There is no mention of mental methods, or decimals, or measures, or rounding and estimating, or converting between units, or the order of operations and the use of brackets, or fractions, or ratio, or geometry, or calculating angles, areas and volumes, or coordinates, or reasoning, or establishing connections, or solving simple word problems or multi-step problems. Instead we are to "focus on larger integers". There is no emphasis on first establishing a robust grasp of place value with three-, four-, and five-digit integers. The link with decimal arithmetic is still largely ignored. And there is no apparent recognition of the need for lots of *word problems* involving measures that might develop the links between number and the focus on solving problems that was declared to be a central part of the underlying 'Aims'. In short, there is no apparent understanding

- that extending the focus prematurely to "larger integers" introduces additional perceptual "noise" and so makes it harder for many pupils to secure their understanding of place value
- that it is better at first to establish a secure grasp of place value etc. in the context of integers with four or five digits (in much the same way as the Singapore curriculum seeks first to establish a robust grasp of addition and subtraction of fractions through extensive work with small denominators ≤ 12)
- that the main reason for moving to "larger integers" derives from the fact that word problems often give rise to *decimal arithmetic* involving numbers with one or two decimal places, which inevitably leads to calculations that involve many more digits (e.g. 47.8×37.6 gives rise to six digits).

We are told that

> pupils in upper Key Stage 2 will be "introduced to the language of algebra".

There is in fact no content listed under 'Algebra' in Year 5—so this development is restricted to Year 6. However, the five content bullet points in Year 6 vary from reasonable, to seriously premature or disturbingly opaque. So the only sensible approach would seem to be for primary schools to view this requirement as preparing the way for Key Stage 3 algebra—and hence to coordinate their interpretation of what is appropriate with local secondary schools (since misconceptions engendered in Key Stage 2 will cause serious trouble later).

It is perfectly reasonable to introduce symbols in a careful and limited way in Year 6. For example, familiar formulae (such as 'area of rectangle = length $\times$ breadth', or 'circumference of circle = $\pi \times$ diameter') can usefully be written in symbolic form ($A = l \times b$; $C = 2\pi r$). One may do much the same to capture the construction rule for a given sequence '$2, 4, 6, 8, 10, \ldots$' by writing the n^{th} term as $2n$. But one has to hope that schools will hesitate to go very much further.

What is clear is that primary pupils need to tackle many more numerical word problems and missing number problems *without algebra* before anyone confuses them by

> "expressing missing number problems algebraically",

or by imposing the possibly important, but poorly expressed requirement that

> "pupils should be taught to enumerate possibilities of combinations of two variables".

We are told that algebra is

> "a means for solving a variety of problems".

In fact much more is true: competence in algebra holds the key to progress both in secondary school mathematics, and in using mathematics in other subjects. Hence it is important to prepare the ground carefully before we introduce this remarkable tool. However, three of the most important prerequisites are missing at primary level.

- Before pupils can appreciate the power of algebra at Key Stage 3, they need extensive experience of tackling numerically based *word problems* using arithmetic (and inverse arithmetic). Hence this needs to be part of pupils' everyday Key Stage 2 experience (as it is in countries from whom we have much to learn, such as Russia, Flemish Belgium, or Singapore). Yet numerical word problems are almost invisible in the statutory programmes of study.

- Second, early arithmetic tends to focus exclusively on calculating "answers". However, as Key Stage 2 pupils meet larger and larger numbers, it should become clear that arithmetic can no longer focus on simply grinding out answers to longer and longer sums. This realisation will be reflected at Key Stage 3 in the shift from *arithmetic* to *algebra*, and this impending shift needs to be pre-figured at Key Stage 2. But how?

  The game of algebra is different from elementary arithmetic. There are no "answers" in algebra. Instead expressions are changed and *simplified using the rules of arithmetic*. That is, algebra is a kind of "structural arithmetic with symbols", which applies the laws of arithmetic to symbolic expressions. This impending shift from the security of calculating answers to the more subtle world of appreciating and exploiting structure needs to be experienced first in the realm of number—before introducing symbols. That is one reason why the principal focus of Key Stage 2 mathematics should **not** be about "calculating with ever larger numbers", but with shifting the focus of work with number

  > away from merely *calculating answers*, and onto *recognising and exploiting* **structure**.

  Yet there is no recognition of this in these programmes of study.
- Third, most non-specialist primary teachers are in no position to teach algebra. In particular, they are in a far worse position to teach this material than specialist Key Stage 3 teachers—who (according to TIMSS and ICCAMS) currently do a very poor job of teaching algebra.

In short, we would do well to interpret this requirement for pupils to be taught "algebra" at Key Stage 2 in a carefully restricted way, by focusing at Key Stage 2 on:

(a) a preparatory numerical treatment of traditional 'word problems';
(b) a limited use of symbols to summarise facts as formulae; and
(c) a clear shift in focus from blind calculation to "structural arithmetic".

The introduction does not improve. We are told that

> "by the end of year 6 [all] pupils will be fluent in [...] long division"

and in working with fractions, decimals and percentages—all of which are themes where the level of English incompetence has been embarrassingly documented in pupils entering Key Stage 4. **All of this material is repeated to some extent in Key Stage 3**. There is therefore

- no need for this material in upper Key Stage 2
- where pupils are unlikely to understand it
- where teachers are unlikely to feel comfortable with it or know how to teach it
- where its imposition would often undermine the principle of "readiness to progress", and
- where there is masses of valuable prior numerical work that needs to be done.

So one has to hope that English pragmatism will prevail and primary teachers will be free to respect the general principles laid out at the very beginning of the document.

Just in case the reader thinks it is safe to read on, the introduction ends with its standard summary of the link between mathematics and language in the last two years of primary school:

> "pupils should read, spell and pronounce mathematical vocabulary correctly."

This is in some sense a desirable goal. But it fails to reflect the profound nature, and the importance, of, the link between language and mathematics at this level.

The content listed for Years 5 and 6 should build on the foundations laid in previous years to tackle more interesting topics and problems. The current programmes of study fail to capture this transition, and need to be revised at the earliest opportunity. These shortcomings are so severe that it is often difficult to comment constructively. Hence comments on the content for 'upper Key Stage 2' are restricted to instances where something useful can be said, and their limited number should not be misconstrued as indicating general satisfaction.

Number—number and place value. The section still shows no appreciation of the challenge of extending pupils' world of number to decimals (for example, to ensure that they make sense of transition at boundaries—when moving from 1.09 to 1.1, or from 2.19 to 2.2, or from 3.99 to 4, or from 9.99 to 10).

Number—addition and subtraction. The neglect of decimals continues—in arithmetic, in rounding, and in solving problems.

Notes and guidance. The level of "guidance" needs to be more profound than repeated emphasis on

> "practise ... to aid fluency".

Number—multiplication and division. Several of the items listed here would seem to be both premature and unnecessary, and may well be better left until Key Stage 3—where they should be better taught, and would actually be used and so be seen to be useful (e.g. "the notation for squared (^{2}) and cubed (^{3})").

The first bullet point mentions "multiples and factors, factor pairs, common factors" but remains silent on the mathematical goal to which these all contribute, namely **factorisation**—the idea that integers can be broken down as the product of smaller numbers, and in particular as a product of prime numbers.

After the prolonged silence on the delicate subject of introducing decimal arithmetic, the seventh bullet point suddenly inserts

> "multiplication and division of numbers involving decimals by 10, 100, and 1000".

The didactical context within which this requirement sits remains totally opaque.

The tenth bullet point announces (in Year 5) that

> "pupils should be taught to understand the meaning of the equals sign".

This cannot mean quite what it says, since the equals sign was introduced in the very first bullet point of "addition and subtraction" back in Year 1. So it needs rewording—perhaps as:

"pupils use = to link equivalent expressions, rather than just to announce an answer."

The final bullet point reveals a profound didactical blindspot. To its credit, the item recognises the need to work with "simple rates". This is an important theme—more important than the programme of study appreciates. The ramifications are extensive, but would seem to have been completely overlooked.

The "simple rates" that are relevant at this stage include things like 'speed' (in metres per second, or kilometres per hour, or miles per hour) and 'prices' (in pounds per metre for cloth, or pounds per kilogramme for heavy goods). Thus 'rates' are by their nature **compound** units. This has two profound implications which appear not to have been recognised.

First, younger pupils access, and make sense of, simple units (such as length, or weight, or money) *directly*. Simple units can be physically handled, and can be directly replicated and subdivided. And given suitably structured practical work, the physical meaning of such units, and the way they combine, can be seen to link fairly naturally with number, so that the addition and subtraction of the associated quantities can be seen to behave like the arithmetic of numbers.

In contrast, compound units are *mental constructions* and cannot be accessed directly. They therefore require rather careful teaching if they are to be understood and handled intelligently. In particular, if 'rates' are needed in Year 5, then some serious preparatory work would seem to be needed **in Year 4**. However, the 'Measurement' headings for Years 1–5 *omit all mention of compound units*, of the need for them to be introduced in advance of their use, and of the difficulties of helping pupils to understand what they mean and how they are used. There is no hint of the need to plan pupils' progression from making sense of simple units to working confidently with compound units. Even in Year 6, the programme of study fails to introduce simple compound units (e.g. speed, rates), and the 'Notes and guidance' mention 'compound units for speed', but only as a possible option.

The second oversight is even more striking. It is well-known that the serious, and more demanding, uses of arithmetic begin as soon as *multiplicative* ideas come into play (e.g. this is one of the central ideas behind the recent ICCAMS study). And despite the lack of any didactical framework for the programmes of study, there is a consistent emphasis on "solving problems" (even if there is little indication as to what sort of problems, and how pupils are to be taught to solve them). Yet there is no apparent understanding here that

problems involving multiplication automatically involve compound units.

How far do we travel if we drive at 30 mph for 2 hours? What is the price of 3kg of beef at £12 per kg? One may choose not to make a fuss about the subtle way the units combine, but one cannot begin to engage with problems that involve multiplication in the absence of compound units. Even the simplest problems such as

"3 children with 4 sweets per child: How many sweets?"

involve the hidden compound unit of "sweets per child". Hence the failure to prepare the ground by introducing compound units would appear to be an oversight.

The reference to "simple fractions" in the final paragraph seems out of place.

Either it does not belong here, or the whole section needs to be revised to make it clear that "number" now includes arithmetic with fractions and decimals.

Notes and guidance. The third paragraph is disturbing. Having just declared, as part of the statutory content, that

> "pupils should be taught to understand the meaning of the equals sign"

(to indicate equivalence), we are now faced with the howler

$$\text{“}\tfrac{98}{4} = 24 \text{ r } 2\text{”}.$$

Such things may be found helpful at some stage; but that stage must come to an end well before pupils are exhorted to "understand the meaning of the equals sign" and to use it "correctly". (If the notation "24 r 2" is found helpful in the early stages of division, it should be clear that one cannot combine it with the = sign in this way—since one could then also write "$\frac{74}{3} = 24$ r 2" even though the two left hand sides $\frac{98}{4}$ and $\frac{74}{3}$ are definitely not equal.) The truth is that the notation "24 r 2" has no part in mathematics, and should by this stage have been replaced by the correct formulation using mixed fractions.

The fourth paragraph tries to say too many things in a single sentence and needs to be revised.

The fifth paragraph (like much else) looks like a note made at some stage that never got edited. It should not use the D-word without clarifying that this is intended for teachers only. The algebraic formulation sits uncomfortably alongside the crass innumeracy of the third paragraph. And it is unclear which of the statutory requirements it refers to. (Could it be the reference to long multiplication? If so, what it is trying to indicate needs to be spelled out more clearly—perhaps as:

> "Short and long multiplication, such as 456×7, exploit two ideas:
> (i) first we partition the number to be multiplied ($456 = 4$ hundreds $+ 5$ tens $+ 6$ units)
> (ii) then we multiply out the brackets, $(400 + 50 + 6) \times 7 = 400 \times 7 + 50 \times 7 + 6 \times 7$, working out each product in turn before combining the results.")

Number—fractions (including decimals and percentages). The first bullet point might make sense if this activity had been begun in Year 4 rather than Year 3, and was now being extended (as one hopes might be the case when things are revised). But it is deeply dispiriting to see the activity beginning too early in Year 3 with

> "compare and order ... fractions with the same denominator"

and taking three full years to progress to

> "compare and order fractions whose denominators are all multiples of the same number".

The meaning of the second bullet point is unclear—partly because of its grammatical abuse: one cannot have

> "equivalent fractions of a given fraction",

so it should perhaps read

> "identify, name and write fractions which are equivalent to a given fraction".

The third bullet point is again ungrammatical (one cannot have "mathematical statements > 1").

There would appear to be a consensus that "comparing and ordering" comes one stage before "adding and subtracting". So if the first bullet point is in Year 5, then the fourth bullet point might be better delayed until Year 6—or at least restricted in Year 5 to fractions whose sum is less than 1, so that the parts can be pictured as being within one whole.

The tenth bullet point is again illiterate and should read:

> "solve problems involving numbers with up to three decimal places."

The eleventh bullet point again contradicts the injunction to "understand the meaning of the equals sign". 50% is an operator, or function; it cannot stand alone, and it is **not** a number. "50% of" has the same meaning as "$\frac{50}{100}$ of"; but 50% is **not** the same as $\frac{50}{100}$.

Notes and guidance. The first paragraph makes a common, but serious, error in that it uses the word "proportions" in its colloquial sense (of "portions" or "parts")—which is not its mathematical meaning. Two quantities may vary together—such as the number of litres of petrol I put in my tank and the amount of money I expect to have to pay. The two quantities are said to be "proportional" if, when one quantity doubles, the other also doubles, and so on: that is, if given any two amounts of petrol—say a litres and b litres—and the corresponding amounts that I have to pay—namely £c and £d—then the ratio $a : b$ is equal to the ratio $c : d$.

The second paragraph rightly draws attention to the need for

> "pupils [to] connect multiplication by a fraction to using fractions as operators ... and to division".

However, to repeat such offhand assertions about what pupils will (or mostly will not) manage to do, while consistently failing to address the missing didactical framework which might help to establish the required "connections", is irresponsible.

The third paragraph again confuses work with fractions and mention of "division with remainders". (The Singapore curriculum restricts "division with remainder" to very early work with integers, and recognises its limitations by abandoning it in upper primary where the focus shifts to working correctly with fractions.)

Measurement. It is not easy to make sense of this listing. The misuse of the word "rectilinear" in the third bullet point needs to be corrected. The reference to volume in the fifth bullet point seems to come out of nowhere (there has been no evident introduction to the awkward idea of volume). And the lack of reference to compound units needs to be reconsidered.

Notes and guidance. The second paragraph attempts a gratuitous use of algebra in a setting where it delivers nothing. The third paragraph is ungrammatical and should have the word "the" deleted.

Geometry. The word "identify" in the first bullet point comes across as awkwardly passive.

In the third bullet point one would prefer to see some more purposeful activity—such as drawing triangles and other figures with given data (for sides and angles).

The fifth bullet point is completely inscrutable. Pupils should indeed be beginning to use properties of rectangles and triangles—but which properties, and how they are expected to be used, remains totally opaque.

The sixth bullet point repeats the *faux pas* of inventing the name "irregular" for polygons which happen not to be regular. It remains unclear what "reasoning" is anticipated—so that one is left with the sense that these words may be mere 'flannel'.

The determined separation of "properties of shapes" and "position and direction" (about which neither the statutory content not the 'Notes and guidance' have anything much to say) looks increasingly irrelevant. The content and structure of the requirements for 'Geometry' clearly need serious re-thinking, and this distinction does not help.

Notes and guidance. In the first paragraph it would be good to see some expected level of accuracy in using a protractor.

The second paragraph has no obvious meaning.

The third paragraph refers to "pupils use angle sum facts"—yet there is no evidence elsewhere of any such "facts" that pupils might be in a position to use.

In the confused 'Note' on 'position and direction', the words "in a variety of diagrams" would seem to convey no useful information. The second 'sentence' has no verb.

Statistics. The first bullet point copies the words from Year 4 and extends them to "line graphs", but it is not clear exactly what this is supposed to mean, or why it belongs under the heading 'Statistics'.

The second bullet point is welcome—though some of this might perhaps be addressed before Year 5, and one would like to see the requirement for pupils to *compile* tables from information given in other ways.

Notes and guidance. It is unclear what is meant by a "time graph".

5.4. *Upper Key Stage* 2—*Year* 6

Number—number and place value. We note once more

(a) the confusion between what constitutes genuine mathematical progress and simply expecting pupils to work with "ever larger numbers",
(b) the references to negative numbers without any clue as to how these are to be presented or understood, and
(c) the lack of attention to decimal place value and decimal arithmetic.

Notes and guidance. The limpness of the additional 'Notes and guidance' speaks for itself.

Number—addition and subtraction, multiplication and division. The amalgamation of the two headings seems sensible—but might perhaps have been introduced already in Year 5.

The demand that all pupils master long division would be wishful thinking at any stage—but is especially so in Year 6. The suggestion that the output from division needs to be interpreted "as appropriate for the context" would be welcome had one not had to comment on the inappropriate interpretation "$\frac{98}{4} = 24$ r 2" in the Year 5 programme of study.

The grammar of the fifth bullet point confuses terms that relate to pairs of numbers (e.g. "common factors"), and terms that stand alone (e.g. "prime numbers"). It needs to be reworded and reordered as:

> "identify prime numbers and find common factors and common multiples"

The seventh bullet point is also ungrammatical and should end:

> "deciding which operations and methods to use and being able to explain why".

Despite the amalgamated section heading, there is a regrettable silence about the need to shift the focus from blind calculation to 'structural arithmetic' (i.e. that pupils should learn to simplify numerical expressions such as

(a) 375×16 by re-bracketing as $3 \times (125 \times 8) \times 2$ so as to exploit the base 10 structure of our numeral system, and
(b) $3 \times 17 + 7 \times 17$ by grouping—or using the distributive law—rather than blindly evaluating).

That is, we need to see a shift towards simplifying calculations by using the laws of arithmetic in concrete form.

Notes and guidance. The first two paragraphs reveal the same old preoccupations with practise and with large numbers.

The third paragraph repeats the howler of "calculating mathematical statements" (with yet another reference to "fluency").

The sixth paragraph uses a curiously permissive linguistic construction. What needs to be said (as becomes clear in the first bullet point of the next section) is:

> "common factors should be related to simplifying equivalent fractions".

Number—fractions (including decimals and percentages). The first bullet point ("in the same denomination") is illiterate; it should read

> "with the same denominators".

The fourth bullet point highlights the absence of any didactical framework within which one might make sense of "multiplying fractions".

The sixth bullet point is again assertive. There is no guide as to how the link between fractions and division can be established (nor is there any recognition that many able pupils emerge from Key Stage 4 without appreciating this link). But even if the link were established in simple cases (such as $\frac{12}{3}$) , there is no recognition of the difficulty of extending this to $\frac{3}{8}$, where a magic wand is waved, a decimal point and some extra 0s are inserted, and the division process somehow continues beyond the decimal point.

The ninth decimal point further undermines the reader's confidence by restricting division to cases where the answer has "up to two decimal places"—having apparently already forgotten the sixth bullet point, where we were instructed to go beyond this with $\frac{3}{8} = 0.375$.

Notes and guidance. The extent of the 'Notes and guidance' would seem to reflect an awareness that support is needed. But the support that is offered needs to be much more carefully crafted. For example, the second paragraph appears to encourage an opportunistic approach to multiplying fractions—by choosing whatever image justifies the rule to be imposed. The third paragraph belatedly recognises that the arithmetic of fractions depends on a consistent treatment in terms of unit fractions—though it comes too late and too cryptically to have the desired effect.

Ratio and proportion. The heading is new in Year 6, yet the section includes no relevant content to be taught. Instead the section is used as a collecting point for various types of problems.

Notes and guidance. The first paragraph seems frighteningly ambitious, but makes a kind of sense as long as it is handled lightly (as is suggested by the final sentence of the fourth paragraph). But when we come to Key Stage 3, there is no sign of the promised "formal approach".

Algebra. As previously mentioned, the five content bullet points in Year 6 vary from reasonable, to seriously premature, or disturbingly opaque.

The first bullet point is fine: it is perfectly reasonable to introduce symbols in a careful and limited way in Year 6. For example, familiar formulae (such as 'area of rectangle = length × breadth', or 'circumference of circle = $\pi\times$ diameter') can usefully be written in symbolic form ($A = l \times b$; $C = 2\pi r$).

The second bullet point might also be fine—provided it is interpreted with a light touch. One would quite like to see pupils looking for the simplest rule to extend and to describe simple sequences (such as $2, 4, 6, 8, \ldots$, or $1, 3, 5, 7, \ldots$, or $4, 7, 10, 13, \ldots$), and to generate sequences (such as $4, 7, 10, 13, \ldots$) for the number of matches in a sequence of configurations (here a 1 by n rectangular string of $n = 1, 2, 3, 4,$ squares with each side made of one match). One may even go so far as to invite pupils to capture the construction rule for a given sequence '$2, 4, 6, 8, 10, \ldots$' by writing the n^{th} term as $2n$. But there are good reasons for hoping that primary schools will not be expected to go much further than this.

The third bullet point begins to go haywire. "Missing number problems" have occasionally been mentioned, but have never been explained. They can be used to strengthen pupils' mental arithmetic and mental flexibility, by requiring them to *work backwards*, but they do not constitute a mathematical topic. When we ask pupils to think about:

> "I double a number and add two and the result is 14. What was my number?"

we are not really interested in getting the answer—but in requiring pupils to work backwards in their heads in preparation for working structurally with unknown numbers. The processes developed in the mind are proto-algebraic; but that does not mean one would be better off introducing a letter for the unknown. Pupils need extensive and planned experience with thinking about such problems *before* the blunderbuss of elementary algebra is introduced to solve them all at a stroke. In particular, **nothing is gained** by prematurely

> "expressing missing number problems algebraically".

The fourth bullet point would be excellent in a context where "an equation in two unknowns" was a natural entity to explore. Yet even in Year 8 or 9, where such equations arise naturally, this requirement is routinely ignored, so that even relatively able pupils do not seem to understand that an equation with two unknowns is a succinct way of capturing an infinite table of pairs of values (or coordinates)—where the values are in no way restricted to being integers. Given that Key Stage 3 teachers routinely fail to address this matter effectively, it is probably premature to impose the requirement in Year 6.

The requirement in the fifth bullet point that

> "pupils should be taught to enumerate possibilities of combinations of two variables"

has no obvious meaning and should either be clarified or deleted. (It may be an attempt to refer to the 'product rule' for systematic counting.)

Notes and guidance. The 'Notes' fail to notice that neither pupils nor teachers are likely to find "symbols", "letters", "variables" and "unknowns" enlightening. Hence the encouragement to introduce their use looks like a recipe for abuse.

Measurement. The continued silence on compound units is embarrassing. It is difficult to comment on those items that are listed in the absence of any evident structure within which they might be taught, understood, and developed.

Notes and guidance. As explained earlier, the third paragraph makes an unfortunate choice (because temperature is not an appropriate measure within which to exercise calculation with units).

The final paragraph not only fails to notice that compound units, such as speed, should have been introduced in Year 4, but gives the impression that compound units remain optional in Year 6.

Geometry. Again, there is a difficulty in commenting constructively because the topics listed are presented in isolation—with no attempt to indicate how they are to be introduced or linked together in a coherent way.

The first bullet point is excellent—but several years too late.

The second bullet point confuses the theme of working in 3D by including "nets", which are of no significance in elementary mathematics. In the days before commercial products such as POLYDRON and ZOMETOOL, nets were essential to allow pupils to *make* their own polyhedra—and they can still be useful. But there is no relevant mathematics of "nets", so they no more deserve mention than do pencils, or graph paper. The important emphasis should be to engage in making and exploring 3D polyhedra, not on any particular way of making them.

The third bullet point fails to mention the need to teach pupils about the sum of the angles in a triangle and to extend this result to quadrilaterals, etc.

The requirement to work in all four quadrants (like the requirement that pupils work with negative numbers) may well prove premature.

Notes and guidance. The attempt in the second paragraph to express coordinates symbolically seems singularly premature—and probably counter-productive.

Statistics. The list seems rather weak.

Notes and guidance. It is hard to see why the third paragraph is included under "Statistics".

6. *Key Stage 3: Programme of Study*

6.1. *Introduction*

The Introduction is mostly excellent: it emphasises the importance of "connections"; it repeats the need for decisions about progression to be based on pupils' "readiness to progress"; and it encourages extension-through-depth as opposed to acceleration. However:

(i) the ensuing list of content then needs to make clear what are seen to be the most important "connections";
(ii) "readiness to progress" should be clarified as referring to having mastered what has previously been taught (rather than waiting for some passive pseudo-Piagetian developmental stage); and
(iii) if extension-through-depth is to be understood, there is an urgent need for detailed explanation of what this means (as proposed in **Recommendation 1** of the ACME report *Raising the bar*[†]).

6.2. *Working mathematically*

The three subsections of the opening section are mostly well-worded, and reflect the three GCSE Assessment Objectives. Given the general nature of the listed requirements it is not easy to identify what may be missing. So I shall simply highlight a few bullet points whose meaning is unclear, or whose wording needs to be improved.

[†]ACME. Raising the bar: developing able young mathematicians, 2012. `http://www.acme-uk.org/media/10498/20121212_acme_raising_the_bar.pdf.pdf`.

(i) Under 'Develop fluency' the third bullet point is poorly worded. Algebra does not "generalise the structure of arithmetic": it retains exactly the same arithmetic as for numbers, but *extends the domain of applicability* to expressions involving letters.

(ii) Under 'Solve problems' the reference in the second bullet to "financial mathematics" has no clear meaning (even that latter-day oracle GOOGLE has trouble finding enlightening references which might explain what these words could possibly mean at lower secondary level). Hence the inclusion of these words can only reflect enforced ritual obeisance to appease some political lobby group. This is wrong, and suggests that we have learned nothing from the "functional mathematics" fiasco. Problems and exercises at this level are bound to include word problems involving money, and work involving percentages (e.g. for discounts, surcharges, VAT, interest). There is no need to invent a new label for this kind of task. And it is deeply disturbing to see evidence of the kind of pressures that may have distorted the drafting process. One suspects that no other developed country would fall into this error (for example, it is easy to check that the Singapore secondary curriculum† does not mention the word "financial").

6.3. *Subject content*

Unlike the curriculum for Key Stages 1 and 2, the curriculum for Key Stage 3 is no longer delineated by year. Nor does it include as much detail as for Key Stages 1 and 2. This may reflect the fact that much of the available drafting effort was devoted to the primary phases, so that Key Stage 3 received less attention.

One unfortunate consequence is that

> the Key Stage 3 curriculum has, as yet, no accompanying 'Notes and guidance'.

This lack of support runs counter to the last two triennial Ofsted reports on mathematics (in 2008 and in 2012), which both emphasised the weakness of mathematics teaching at this level.

If one ignores the lack of detail, then the listed content is, on the whole, reasonably well-chosen. However, the compressed style makes it even more disturbing that the essential underlying didactical thrust is never made explicit, and is left to be somehow inferred from the listed themes. This seems likely to lead to teachers and textbooks 'covering' the listed themes without thinking what preliminary work is needed, without making the important connections, and without laying the foundations for mathematical reasoning that will be needed at Key Stage 4 and beyond.

What is missing goes well beyond the stated desire to "avoid excessive detail"—as is illustrated by the breath-taking silence with regard to how arithmetic is to be extended to include *negative numbers*. Moreover, many of the important connections that could easily have been specified are also left to be inferred.

Given that the 'Notes and guidance' consistently declare themselves to be "(non-

†Ministry of Education, Singapore, Secondary Mathematics Syllabuses, 2007. `http://www.moe.gov.sg/education/syllabuses/sciences/files/maths-secondary.pdf`.

statutory)", one has to hope that this missing (non-statutory) support will be provided in the coming months—after the publication of the statutory curriculum, but in time to help schools when they come to draft their own obligatory year-by-year programmes of study.

Number. Four points stand out—three relatively minor and one more serious.

In the third bullet point it is unclear what is meant by "product notation". If this refers to writing an integer as 'a product of prime powers', then it would be more helpful to say precisely this. In general the explicit mention of "unique factorisation" is welcome—for this is a fundamental property of integers which has never received its due attention in English secondary schools. However, given this historical lacuna, it will not be clear to most teachers what "uses" this property can be put to as part of Key Stage 3—so suitable 'Notes and guidance' will be needed.

The reference in the fourth bullet point to "proper and improper fractions" and to "mixed numbers" would seem to be a peculiarly English disease. These terms make sense in primary school, where fractions are at first "parts of a whole", and fractions larger than 1 are therefore thought of as being "improper". But at secondary level a fraction is a fraction— whether less than 1 or greater than 1—and the expression "improper fraction" ceases to have a place. Exactly the same is true of "mixed numbers": it may still sometimes make sense to give an answer in the form $1\frac{1}{2}$, but that does not mean that "mixed numbers" deserve explicit mention in the secondary curriculum. (One suspects that this is understood in other countries: for example, the words "improper" and "mixed fractions" do not occur in the Singapore secondary curriculum.)

The third last bullet point includes a curious use of inequalities. In one sense there is no right or wrong here, but if one is to use inequalities to specify *errors*, then—given the convention with regard to rounding decimals—one would expect to see the two inequality signs *the other way round*, so it should probably read:

> "using inequality notation $a \leq x < b$".

The last bullet point is rather more serious. We are told that:

> "Pupils should be taught to appreciate the infinite nature of the sets of integers, real, and rational numbers".

It is hard to imagine what exactly is expected to be *taught*, or why it has been included at this level. It is also unclear why more care was not taken over the wording: Why do we find the sudden appeal to the language of "sets"? What are these "real" numbers that suddenly appear? And why do "reals" come before "rationals"?.

Infinity has fascinated young minds for centuries; but it is a *background* theme rather than a curriculum topic. Counting numbers go on for ever—as do the decimal places in a recurring decimal; but these are not good reasons for drafting woolly curriculum requirements such as the one above. Later (but scarcely at Key Stage 3) one might infer the existence of irrationals from the fact that the decimal of any rational 'recurs' (so if we can explain why

$$0.012345678910111213141516171819202122232425\ldots$$

or

$$0.1001000100001000001000000100000001000000001 0\ldots$$

do not recur, then they cannot represent rational numbers). One might also enjoy proving that the prime numbers "are more than any assigned multitude". But "the infinite nature of the integers and reals" will still remain elusive.

It would be far better to specify clearly what pupils are expected to encounter and to be able to use, and to avoid pretentious language such as occurs in this bullet point—especially at Key Stage 3.

Algebra. The topics listed in this section mostly make sense, though the didactical architecture that might bring them to life remains implicit (or painfully suppressed—as in the intended treatment of negative numbers, of which there is not a single mention). In particular, it needs to be made clear how the words used in each bullet point can be interpreted at different levels, so that teachers can respect the general advice at the beginning of the *Introduction* that

> "those who are not sufficiently fluent" should "consolidate their understanding before moving on".

Over-ambition spills over in the fourth last bullet point. Here the curriculum confuses "what an individual teacher is free to include if s/he wishes" with what is statutorily required of *all* pupils. One can see reasons why Key Stage 4 might include the modulus function, and exponential and reciprocal graphs; but even at that level it might be hard to justify including "piece-wise linear graphs". (The Singapore secondary curriculum does not mention "piece-wise linear" even at that level.) To require all pupils to work with

> "graphs of a variety of functions including piece-wise linear, exponential and reciprocal graphs"

at Key Stage 3 (when their idea of a "function" extends no further than the simplest algebraic expressions) would seem to be excessive. (One can imagine particular exercises which lead to a graph which is "piece-wise linear"—but one cannot expect the majority of Key Stage 3 pupils to perceive, or to interpret, such a graph as being the graph of a 'piece-wise linear function'.)

The final bullet point may be an example where a preliminary draft survived without being revised. "Appreciate other sequences that arise" is far too vague. Perhaps it meant to say something like,

> "recognise simple geometric sequences, and be prepared to analyse other sequences—whether given numerically, or arising from counting".

Ratio, proportion and rates of change. Most of the statements under this heading make reasonable sense—though again there is a danger that the topics will be taught in isolation.

In the first bullet point, changing units should include simple compound measures (such as speed).

The third bullet point is poorly worded, and should read:

> “where the fraction may be less than one or greater than 1” (i.e. “or”, not “and”).

The third last bullet point is another giveaway. The statement would be far clearer, and everyone would understand it, if it finished after “and simple interest”. The words “in financial mathematics” are redundant, have no clear meaning, and were obviously included to appease some lobby group. That is wrong.

Geometry and measures. For the last 30 years there has been no serious professional debate about what should constitute school geometry. So it is not surprising that this section lacks coherence, and includes the occasional ‘howler’.

The only omission that leaps out from the page is the complete silence on isosceles triangles. And though it is refreshing to see that, unlike previous versions of the national curriculum, Pythagoras’ Theorem is here expected to be “derived”, this would make educational and mathematical sense only if it is to be “derived” from simpler, known results. Yet the programme of study fails to reveal any clear framework of simpler results from which the result might be “derived” in this sense.

In the fifth bullet point the final “and” should almost certainly be an “or”.

The seventh bullet point is probably where reference to “isosceles triangles” should be inserted. The parenthetical remark “[for example, equal lengths and equal angles]” is meaningless and looks like a ‘note to self’ that never got revised as intended. It should probably read something like:

> “derive and illustrate properties of triangles, quadrilaterals, circles, and other plane figures [e.g. proving that the base angles of an isosceles triangle are equal] using appropriate language”.

The ninth bullet point needs rewording. It is possible to “identify (two) congruent triangles” in a given figure; but one never “constructs (two) congruent triangles”. One suspects that what was meant was:

> “identify congruent triangles in a given figure, and construct a triangle from given data (SSS, SAS, or ASA)”.

Probability and Statistics. The themes listed have the virtue of not being excessively ambitious. But the lack of detail is perhaps more serious here than elsewhere—for there is no longstanding tradition which might help teachers to interpret what is expected. There is no discussion of how to collect data; no mention of bias; no discussion of sampling; there is an injunction to compare observed distributions, but no comparison of different modes of representing given data; no mention of quartiles, deciles, or percentiles; no mention of interquartile range; no discussion of what probability is, or of what this tells us about calculating the probability of compound events.

One is left to hope that textbooks and examiners will not simply revert to the same old interpretation, but will use the implied freedom wisely. And it is essential that government supports sensible pilot projects in this area, so that when we come to revise the current curriculum, we will be better placed to know what material in probability and statistics should be included, at what stage, and in what form.

About the author

Tony Gardiner (born 1947) is a British mathematician. He was responsible for the foundation of the *United Kingdom Mathematics Trust* in 1996, one of the UK's largest mathematics enrichment programs, initiating the Intermediate and Junior Mathematical Challenges, creating the *Problem Solving Journal* for secondary school students and organising numerous masterclasses, summer schools and educational conferences. Gardiner has contributed to many educational articles and internationally circulated educational pamphlets. As well as his involvement with mathematics education, Gardiner has also made contributions to the areas of infinite groups, finite groups, graph theory, and algebraic combinatorics.

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