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THE METHODOLOGY OF MATHEMATICS

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1. *Introduction*

This essay is based on a talk given by the first author to students and staff of the Departmento de Geometría e Topología at the University of Seville in November, 1993. The issues presented there were part of a continued debate and discussion at Bangor over many years, and this explains why this is a joint paper.

The aim of the talk, and the reason for discussing these topics, was to give students an understanding and a sense of pride in the aims and achievements of their subject, and so help them explain these aims and achievements to their friends and relatives. This pride in itself would be expected to contribute to their enjoyment of the subject, whatever their own level of achievement. Because of this, and because of its origin, the tone of the article is principally that of an address to students.

We do not claim to be alone in addressing these questions. Dr Allan Muir at City University organised a "How Mathematics works" group for some years, and there are similar groups in, at least, the U.S.A. Many of these issues are discussed in the books by Davis and Hersh [7, 8].

We start with some general questions to which we believe it is helpful for students to be able to formulate some kind of answers. The question for teachers of mathematics at all levels is to what extent, if at all, the training of mathematicians should involve professional discussion of, and assessment in, possible answers to these questions, such as those given or suggested here[†].

2. *Some basic issues for mathematicians*

(1) Is mathematics important? If so, for what, in what contexts, and why?
(2) What is the nature of mathematics, in comparison with other subjects?
(3) What are the objects of study of mathematics?
(4) What is the methodology of mathematics, what is the way it goes about its job?
(5) Is there research going on in mathematics? If so, how much? What are its broad aims or main aims? What are its most important achievements? How does one go about doing mathematical research?
(6) What is good mathematics?

It may be thought by some that these questions are beside the point, a waste of time, and not what a real mathematician should be considering. Against this we would like to give a quotation from Albert Einstein (1916) [9]:

> How does a normally talented research scientist come to concern himself with the theory of knowledge? Is there not more valuable work to be done in his field? I hear this from many of my professional colleagues; or rather, I sense in the case of many more of them that this is what they feel.
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> I cannot share this opinion. When I think of the ablest students whom I have encountered in teaching – i.e., those who have distinguished themselves by their

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[†]This article has previously appeared in various versions and in various places, [4], but it seems that some of the points discussed could be worth raising with a new group of those most actively concerned with the education of mathematicians and through them with a new generation of students perhaps at all levels, so we are grateful for the opportunity to publish a slightly revised and updated version in the De Morgan Gazette.

> independence and judgement and not only mere agility – I find that they have a concern for the theory of knowledge. They like to start discussions concerning the aims and methods of the sciences, and showed unequivocally by the obstinacy with which they defend their views that this subject seemed important to them.
>
> This is not really astonishing. For when I turn to science not for some superficial reason such as money-making or ambition, and also not (or at least exclusively) for the pleasure of the sport, the delights of brain-athletics, then the following questions must burningly interest me as a disciple of science: What goal will be reached by the science to which I am dedicating myself? To what extent are its general results 'true'? What is essential and what is based only on the accidents of development? [...] Concepts which have proved useful for ordering things easily assume so great an authority over us, that we forget their terrestrial origin and accept them as unalterable facts. They then become labelled as 'conceptual necessities', 'a priori situations', etc. The road of scientific progress is frequently blocked for long periods by such errors.
>
> It is therefore not just an idle game to exercise our ability to analyse familiar concepts, and to demonstrate the conditions on which their justification and usefulness depend, and the way in which these developed, little by little ...

There are a number of reasons apart from authority to consider the above questions. A professor of mathematics in the UK with whom we discussed these questions suggested that the aim of considering them was to get students to reflect on the methods of mathematics. He remarked, as if seeing this for the first time, that there was a well known difference between human beings and other animals, that humans have this ability to reflect on what they do, and that this ability affects beneficially a lot of human activity. One aspect of this reflection is that it leads to the notion of value judgement, again a facility which humans have which is not apparently shared by other animals, or at least not in a way in which we can communicate, by and large.

Reflection on an activity is, generally, a useful way of increasing its effectiveness, as we are able to analyse what is essential, what is important, and how the activity can be done avoiding the easiest of mistakes in method. On these grounds it is reasonable that we should reflect on the activity of mathematics. In reflection, we also usually are aware of the value of the activity.

Another reason for our considering these questions was through a comparison with aspects of education in art.[†] We have heard it argued that education in art and design is considerably ahead of science education in arousing the interest and independence of students, so it is worth considering how these educators go about things. Here are aims which have been given for a course in design:

(1) To teach students the principles of good design.
(2) To encourage independence and creativity.
(3) To give students a range of practical skills so that they can apply the principles of good design in employment

Is there something here from which mathematics courses can learn? Is it reasonable aims for a mathematics course to replace in the above the word "design" by the word "mathematics"? If not, why not? Here is another quotation, from the book by T. Dantzig, [6]:

> This is a book on mathematics: it deals with symbols and form and with the ideas which are back of the symbol or the form.
>
> The author holds that our current school curricula, by stripping mathematics of its cultural content and leaving a bare skeleton of technicalities, have repelled many a fine mind. It is the aim of this book to restore this cultural content and present the evolution of mathematics as the profoundly human story it is.

[†]For a discussion of Art and Mathematics see the site on Symbolic Sculptures and Mathematics at http://groupoids.org.uk/popmath/cpm/sculpmath/sculpmath.html.

Is there something in this from the point of view of a higher level of teaching of mathematics? This book dates from the 1930s. Have we made much progress since then in dealing with the points he raises?

Now let us consider the questions one by one.

3. *What is the importance of mathematics?*

It is not generally recognised how much of a part mathematics plays in our daily lives. Some of the mathematics is of course quite old: every day we use numbers, graphs, addition and multiplication. It is easy to forget that the invention of these was at one time a great discovery. The replacement of Roman numerals by Arabic numerals, and so the possibility of a good bookkeeping system, is said to have led to the prosperity of Venice in the 14th century. It is also of interest here to note the importance of pedantry in mathematics. A key aspect of the Arabic system is its use of the number zero. At first it seems absurd to count the number of objects in an empty box; and at one time emptiness, a vacuum, was associated with evil. The surprise is how essential is zero for an adequate numeration system, in which the number 0 is used as a place marker. The lack of this concept of zero held up the progress of mathematics for centuries.

On a higher level, without the mathematics of error correcting codes we would not have had the beautiful pictures of Jupiter from the Voyager II. This mathematics is also essential in many aspects of telecommunications and of computers, and in particular for CD players and most communication systems since their invention.

There is an amusing story about this last application, [13]: Negotiations between Sony and the Dutch company Philips about the standards for CD were held by top management. The Japanese considered Philip's proposal for error correction inferior to theirs, and in the end the Japanese proposal was accepted. Back in Eindhoven, the embarrassed managers called in their science directors to declare that the company did not have sufficient expertise in this area called "coding theory" and to find out where in Europe the real experts could be found. To their surprise, the answer was: "in Eindhoven!", in the person of the Dutch number theorist Van Lint!

Without the mathematics of cryptography, there would not be possible the current level of electronic financial transactions crossing the world, and involving billions of dollars. Currently, the mathematics of category theory, a theory of mathematical structures, is being used to give new insights into future logics and algebras for the design of the next generation of programs and software. The enormous applications of mathematics in engineering, in statistics, in physics, are common knowledge. It is also imagined that the role of mathematics is being taken over by the use of supercomputers. It is not so generally realised that these supercomputers are the servants of mathematical and conceptual formulations: the electronics is marvellous in that it does the calculations so quickly and accurately.

For another example, body scanners are an application, a realisation, of a piece of 19th century mathematics expressing how to reconstruct a solid object of varying density from views through it, nowadays of an X-ray, where the only measurement is the change of intensity as the ray passes through the body, for a large number of varying positions of the ray. The theories of the big bang, of fundamental particles, would not be possible without mathematics.

4. *What is the nature of mathematics?*

There is here a mystery. The Nobel prize-winner E. Wigner has written a famous essay *The unreasonable effectiveness of mathematics in the natural sciences* [16]. For us, the key word is "unreasonable". He is talking about the surprise that the use of mathematics is able to give predictions which are in accord with experiment to the extent of nine significant figures. How is such astonishing accuracy possible?

It seems likely that a full "explanation" of the success of mathematics would need more understanding of language, of psychology, of the structure of the brain and its action, than is at present conceivable. Even worse, the development of such understanding might need, indeed must need, a new kind and type of mathematics. It is still important to analyse the scope and limitations of mathematics. It is also reasonable that an indication of such an analysis should be a necessary part of the education and assessment of a student of mathematics. Of what use is a student who does not know such things?

Here then are some quotations from this article:

> ... that the enormous usefulness of mathematics in the physical sciences is something bordering on the mysterious, and that there is no rational explanation for it.
>
> Mathematics is the science of skilful operations with concepts and rules invented just for this purpose. [this purpose being the skilful operation ...]
>
> The principal emphasis is on the invention of concepts.
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> The depth of thought which goes into the formation of mathematical concepts is later justified by the skill with which these concepts are used.
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> The statement that the laws of nature are written in the language of mathematics was properly made three hundred years ago; [it is attributed to Galileo] it is now more true than ever before.
>
> The observation which comes closest to an explanation for the mathematical concepts' cropping up in physics which I know is Einstein's statement that the only physical theories which we are willing to accept are the beautiful ones. It stands to argue that the concepts of mathematics, which invite the exercise of so much wit, have the quality of beauty.

It is of interest to compare mathematics with other subjects, and to link this comparison with the question of the objects of study of a subject, and of its importance.

Suppose we ask questions of a few of our fellow scientists as to why one should study and fund their subject. Answers might run as follows:

The astronomer: In astronomy we study the beginnings of the universe, and the flow of time over billions of years, as well as the furthest distances of space. What could be more enthralling? We have some money for this study, with various telescopes over the world, but of course not enough.

The physicist: In physics, we study the fundamental constituents of matter. What could be more fascinating? Without physics, there would be no astronomy, for example. Thus many more of the best students should study physics, and the Government should give us a lot of money.

The chemist: In chemistry, we make molecules do things for us, so that chemistry is part of our everyday lives. Without the understanding found by chemistry, there would no study of the stars, and no understanding of biology, of the formation of the planets. So, many of the brightest students should study chemistry, and the Government should give us a lot more money.

The biologist: Biology is about life. What is life? How did it come about? How does it interact with us and the world? We are all concerned with life. So, many more students should study biology, and the Government should give us a lot more money.

The engineer: Engineering is about making things which control our environment and do things for us. Without engineering, modern civilisation is inconceivable. Many more students should study engineering, and the Government should give us a lot more money.

Of course there are many more protagonists in this story. Also, we have exaggerated the concern with finance. Yet, the financing of an activity is one measure of the importance attached to it.

Let us turn now to the mathematician, and ask for his or her story and justification for existence in the hustle and struggle for a place in the scheme of things. It is quite possible that from even a well known mathematician you will get no clear answer. It might be claimed as an important achievement that, for example, Fermat's last theorem has now been solved. Would such a solution, however, bring in the crowds or the cash? Why should it? Certainly, the solution is a tremendous achievement, but what is its general import? Why was so much effort devoted to it? Is it merely comparable with breaking another record? There are answers to these questions, but the questions seemed not to be asked in the glory of an apparent solution to a longstanding problem.

These questions are not idle. Resources are limited. Any one person's interests are limited. We need a more convincing answer for the support of our subject, and to persuade people to study it. Here is our try:

The mathematician: Mathematics develops language for expression, deduction, validation, calculation. It is about the study of pattern and structure, and the logical analysis and calculation with patterns and structures.

In our search for understanding of the world, driven by the need for survival, and simply for the wish to know what is there, and to make sense of it, we need a science of structure, in the abstract, and a method of knowing what is true, and what is interesting, for these structures. Thus mathematics in the end underlies and is necessary for all these other subjects. This is part of our claim for your attention, and for the support for our studies.

Another part of this claim is the fascination and wonder at the new patterns and structures, the surprising relationships, which our study has found. Mathematics also brings humility. We know how hard it can be to decide the truth of but one apparently simple and clear statement. We are aware of the limitations of mathematical truth, that not all that is true can be proved, as shown by the undecidability results of Godel. You will not find a mathematician writing that the final solution, the unified theory which will solve everything, is at hand. Rather, we are looking for the surprises which show us a new view of the world, and new riches to explore. Experience leads us to expect these to appear. For the mathematician, the world is not only stranger than you imagine, but stranger than you can now imagine. It is our job to investigate this strangeness.

5. *What are the objects of study of mathematics?*

This has already been answered to some extent. Mathematics does not study things, but the relations between things. A description of such a relation is what we mean by a "concept". Thus we talk about the distance between towns, and might feel this is less "real" than the towns themselves. Yet even the word "town" denotes a concept! Nonetheless, relations between things, and our understanding of these relations, is crucial for our operation in and interaction with the world. In this sense, mathematics has the form of a language. It must be supposed that our ability to operate with concepts, with relationships, had and maybe continues to have an evolutionary value.

It is also curious in this respect that the achievements of mathematics are generally held by mathematicians to be the solution of some famous problem. Certainly such a solution will bring to the solver fame and fortune, or at any rate a certain fame within the world of mathematicians. Yet the history of mathematics and its applications shows that it is the language, methods and concepts of mathematics which bring its lasting value and everyday use. We have earlier mentioned some examples of this. At a more advanced level, we can say that without this language, for example that of groups and of Hilbert spaces, fundamental particle physics would be inconceivable.

Some of the great concepts which have been given rigorous treatments through this mathematisation are:

> number, length, area, volume, rate of change, randomness, computation and computability, symmetry, motion, force, energy, curvature, space, continuity, infinity, deduction.

Very often the problem to make some mathematics is, in the words of a master of new concepts, Alexander Grothendieck, "to bring new concepts out of the dark" [11]. It is these new concepts that make the difficult easy, which show us what has to be done, which lead the way.

More important is the way mathematics deals with and defines concepts, by combining them into mathematical structures. These structures, these patterns, show the relations between concepts and their structural behaviour. As said before, the objects of study of mathematics are patterns and structures. These patterns and structures are abstract, a notion discussed below.

6. *What is the methodology of mathematics?*

Here again is a subject which is rarely and not widely studied. There is the comment of Paul Erdős that mathematics is a means of turning coffee into theorems. Perhaps, though, this does not help the beginner too much. So let us look at some of the issues discussed in the books by P. Davis and R. Hersh [7, 8], *The Mathematical Experience*, and *Descartes' Dream*, particularly the section of the first book on “Inner issues”. This deals with a number of themes.

6.1. *Symbols.* The use of symbols and symbolic notations is one of the characteristics of mathematics, and one which puts off the general public. People will say they were able to do mathematics till it got onto x and y. The manipulation of symbols according to rules is still an important part of the craft of mathematics. We find we have to teach people who wish to master say economics but who are unable to deduce from $x + 2 = 4$ that $x = 2$. This makes very difficult the understanding of the concepts of economics. Very complicated relations can be expressed symbolically in a way which can hardly be conveyed in words. This economy which symbols allow is improving continually, as the symbols are used in the denotation of advanced concepts and the rules of the symbol manipulation are used to model the rules for the concepts.

It has been said, in an exaggerated way, that the history of mathematics is the history of improved notation. This reflects the finite nature of intelligence, which requires props and metaphors to help and guide it.

Some symbols are in themselves metaphors. Examples are

$$= , \neq, \exists.\aleph, \forall, \surd, \|, \infty, \emptyset, \to, \leftarrow, <, \leqslant, >, \geqslant,$$

and so on. Others have acquired strong associations, so that we can use them as metaphors. Symbols are able to express “with economy and precision”, to use words of A.N. Whitehead[†]. The use of particular symbols is something that changes with time, as mathematicians become accustomed to and find appropriate a new notation.

In some cases, a notation, brought about by the laziness of mathematicians, leads to a new theory. For example, expressions of the type

$$(a_{11}x_1 + \ldots + a_{1n}x_n, \ \ldots, a_{m1}x_1 + \ \ldots + a_{mn}x_n)$$

get abbreviated over time to Ax, which you can see is simpler to write without understanding what the displayed line means. In order to allow for the correct manipulation of this abbreviation it turns out that the rules for what are called matrices were worked out, and these are widely used in mathematics, science and engineering.

To give an example close to the heart of some of our research, the first author has been concerned since 1965 as to whether the linear notation for mathematics is a necessity or a historical result, based on the needs of printing. The analysis of this linguistic point has led to a new kind of “higher dimensional algebra”, in which symbols are related not just to those to the left and those to the right, but also up and down, or out of the page, as well. This algebra then becomes closer to and more able to model some geometric situations, and this leads to the formulation and proofs of new theorems, to new calculations and insights. A web search on “higher structures in physics” indicates current interest.

6.2. *Abstraction.* This is an essential part of mathematics, and again is one part of what makes mathematics incomprehensible to the general public. As said above, mathematical structures are abstract. They are defined by the relations within them. They are thought of as non-sensual.

The advantages of abstraction are at least threefold.

An abstract theory codifies our knowledge about a number of examples, and so makes it easier to learn their common features. A multiplicity is replaced by one theory. This codification exploits analogies, not between things themselves, but between the behaviour and relations of things. Finding these analogies, such abstraction, is an important method in mathematics.

Once the theory is available, it may be found to apply to new examples. This leads to the excitement and joy of “That reminds me of ...!”. For this new example, a body of established theory is available, at the turn of a page.

An abstract theory allows for simpler proofs. This is a surprise, but is commonly found to be true. The abstract theory allows for the distillation of essentials. It is of interest to know if a theorem or

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[†]The situation is maybe much more subtle, and needs further analysis. For example, the use of the symbol = is actually quite subtle.

fact is true in the general situation or only in the particular example. The abstract theory allows for the removal of possibly irrelevant aspects[‡].

6.3. *Generalisation and extension.* This has some features in common with abstraction, but usually applies differently. Thus a generalisation of the (3,4,5) right angled triangle is Pythagoras' Theorem, while an extension is Fermat's Last Theorem, which says that the equation $x^n + y^n = z^n$ has no solutions for positive integers x, y, z if $n > 2$. This was thought recently to have been settled, but it at first seemed there was still a gap in the proof. (That gap has now been filled.)

6.4. *Proof.* The rigorousness of the notion of proof is a particular feature of mathematics. It is why mathematics is essential in engineering, safety, physics and so on. The notion of proof, of validity, in mathematics, is an aspect of the general question: What is the notion of validity in an area of study? Each area, from social sciences, economics, chemistry, biology, education, law, literature, and so on, has its notion of validity, and the contrast and uses of this notion are of particular interest.

The question of what is acceptable as a valid proof in mathematics is still subject to argument and discussion, particularly with the existence of very long proofs (for example 15,000 pages, [10]), and with the use of computers for visualisation, experimentation, and calculation.

As we were preparing this version of this essay for publication, the death was announced of Vladimir Voevodsky who had made a great contribution to modern algebraic geometry and then began to doubt if all the proofs were 'fool proof'. In the article [15], which is well worth reading in full for other points relevant to our discussion, he relates:

> Only then did I discover that the proof of a key lemma in my paper contained a mistake and that the lemma, as stated, could not be salvaged. Fortunately, I was able to prove a weaker and more complicated lemma, which turned out to be sufficient for all applications. A corrected sequence of arguments was published in 2006.
>
> This story got me scared. Starting from 1993, multiple groups of mathematicians studied my paper at seminars and used it in their work and none of them noticed the mistake. And it clearly was not an accident. *A technical argument by a trusted author, which is hard to check and looks similar to arguments known to be correct, is hardly ever checked in detail.* (Our emphasis)

This, and other similar reflections, for which see the article, led Voevodsky to try to develop new foundations for mathematics, ones that could be double checked by computer. This is current and important. It relates fundamentally to the idea of the nature of proof and to the methodology of mathematics. The point is that the technical nature of the arguments concerned made it extremely difficult to check, so the reader assumes that the writer of the proof has checked everything in great detail. He had, but still had made a slip somewhere.

This raises a separate point on the role of concepts in proof already mentioned with regard to Grothendieck on page 31. The mathematician, Raoul Bott, commented to the first author in 1958: *Grothendieck was prepared to work very hard to make things tautological!*; Grothendieck describes how he would develop a general theory in an attack on a particular hoped for result. The theory was likened to the 'rising sea'. To quote an article by Colin McClarty, [14], himself quoting Grothendieck, from [12, p. 553],

> In this "rising sea" the theorem is "submerged and dissolved" by some more or less vast theory, going well beyond the results originally to be established.

[‡]Now we would also emphasise the relation of abstraction with **analogy**, showing relations between aspects of many examples, and allowing the interaction of seemingly quite different theories. See this discussion: `http://groupoids.org.uk/popmath/cpm/exhib/pagesexhib/analogy.html`.

Thus you generalise and abstract until the searched for result is almost a tautology. The generalisation is not just for its own sake but eventually to *simplify* and to *clarify* the theory.

6.5. *Existence of mathematical objects.* A great mathematician has urged that the major problem of mathematical education is to teach the reality of mathematical objects. What is this reality? In what way do these objects exist?

This question has been a matter of major interest to many philosophers of mathematics, but its interest is perhaps in the process of being downgraded. Mathematics is often about processes. The question of existence of a mathematical structure is maybe like asking whether the game of chess exists. Clearly it does not exist in the way that tables and chairs exist, but none the less, it influences many lives, and passes the cash test. (Does it earn money? The answer is clearly: Yes, for some, for example tournament winners, writers on chess, and makers of chess equipment.)

The relation of mathematical concepts and methods to processes is indicated by the way that memory of muscular action and rhythm are important aspects of mathematical work. A lot of mathematics is concerned with the realisation and understanding of the effect of repetitive processes and methods. Mathematicians are good at understanding and imagining moving things around, such as from one side of an equation to another, or changing a pattern in space. They use movements of their hands and arms to convey what is happening. The objects and ideas of which mathematicians talk are sometimes a kind of concatenation of a variety of such remembe processes. The representation of these ideas in writing is by contrast often bare and sparse, and this is part of the difficulty of learning the use and application of these objects and ideas. On the other hand, it also allows for each person to make the interpretation and internalisation most appropriate to themselves.

6.6. *Infinity.* The taming of the infinite, or the enlargement of the imagination to include infinite operations, is one of the joys of mathematics, and also one of the scandals. Are these infinite objects real? The surprise is that these infinite, possibly unreal, objects can be used to prove finite real things, and this again is an aspect of the mystery of the subject. Suppose for example that these infinite objects are used to prove the safety of a nuclear installation, or of an aircraft landing system? What credence should be placed on such a proof? These are real issues.

7. *Is there research going on in mathematics?*

Those who wish a practical test should look at the change in Mathematical Reviews since it was started in 1940. This monthly journal contains abstracts of mathematical papers. Roughly speaking, a few paragraphs are enough for a five page paper. The growth in terms of numbers of pages over these years is about eleven times. Each month in 1994 when the first version of this essay was written, there were published about 400 large pages of abstracts of mathematical papers[†]. This is indeed the golden age of mathematics, both in quantity and quality.

The aims of this research are at various levels. One is the advancement in knowledge about particular types of structures, which are already well defined. Another is the introduction of the study of new structures, as they have appea and been shown to be relevant. There are new relations between structures. There is the urge to simplification, to find structures which explain structures, and help us to understand the way particular structures behave in themselves and relate to other structures.

What is difficult for the newcomer in the field, and for the general public, to understand is how one goes about doing mathematical research. Here we give some pointers, by suggesting four ways of going about the job, of intention. There are certainly many more, and each individual researcher must in the end devise his or her own strategy for success. It is also difficult to know how much one must know before starting on mathematical research. A famous answer to this particular question was: "Everything, or nothing." (Quoted in E. T. Bell's *Men of Mathematics* [2].)

7.1. *Method 1: Apply a standard method to a standard type of problem.* This has every guarantee of success, provided one is sufficiently skilful in the standard method. This method

[†]In 1996 Mathematical Reviews went on-line and continues to grow. A glance at their home website, http://ams.org/publications/math-reviews/mr-history gives some picture of the growth. In particular, there is now a "reviewer community numbering nearly 17,000 researchers".

is probably a part of every successful research project. Indeed, a common method of mathematical research is to reduce a problem to one already considered. If the original problem is too difficult, then a standard strategy is to simplify the problem so that it does become of standard type, before adding the complications which make it a new problem. The general presumption might be that one can only do easy things. So the method is to reduce a problem to a type that can be seen to be easy. If in doubt, do the obvious thing first.

Those who practice and become skilful at applying standard methods, may someday find that their skills apply to a problem no one else has considered, and that this leads to new and important results.

Much of the education of a mathematician is concerned with acquiring the skills and knowledge appropriate to work in a chosen area.

7.2. *Method* 2*: Attacking a famous problem at the frontiers of knowledge.* This is the strategy of going for a famous problems at a peak of knowledge. The advantage is that if you succeed, then you will become famous. It is more difficult to assess your chances of success. You will probably need some new ideas.

This seems to be the most ambitious method for a young person. However, S. Ulam, in conversation with the first author at a conference in 1964 suggested that while this method might appeal to a young ambitious person, concentration on this might distract her or him from developing the kind of mathematics most appropriate to themselves.

Usually, though, one attacks smaller problems at the frontiers of knowledge, problems to which not so much effort has been devoted, and so where there is a greater likelihood of success. You will almost certainly have to study to find what has been done, what techniques are available, and which you need to master. It is helpful to have problems for which the criteria for success are clear: the answer is yes or no to some question. On the other hand, failure to provide a solution is also clear cut, as is finding the problem too easy. Mathematicians need to build into their strategy plans for dealing with both too little and too much success on the problem at hand.

7.3. *Method* 3*: Relate different areas of knowledge.* In this method you learn about the beginnings of different areas, and find relations between them. So you fill in the gaps between the peaks, while often the "top people" are occupied with building up the peaks. The advantages of this method are that you learn something of different areas, and in a useful way, since you have to work to do the translations between the two areas. This is a good method for PhD theses, since a supervisor can often see the relation without having worked out the details. It also advances the general unity of mathematics. Another advantage is that it gets you used to the idea of proving small but useful results which help to fill in the gaps and create the picture of what is going on.

7.4. *Method* 4*: Blue sky research.* Here you have some idea of a mathematics which ought to exist, and the characteristics of it. You also have a few hints as to the kind of materials of which the mathematics ought to be made. The problem is that proper mathematics requires definitions, examples, propositions, theorems, proofs, calculations, and in the beginning none of these exist. So they have to be assembled over a period of time. In what order should this be done, and how important will the work be? This can hardly be judged till the theory is worked out, and such a theory does not emerge, like Venus Anadyamene, fully formed from the sea. A theory accumulates in a journey over a period of years, and a gut feeling of importance of a line of investigation is necessary to motivate travel on a long road.

We have both been working on this kind of research, as well as other kinds, for decades. The first author formulated the theme of higher dimensional algebra[†] in the mid 1960's. In this algebra, symbols are related not just to those to the left or right on a line, but also to those up or down, or even out of the page. The aim was that of an algebra more closely related to the geometry, and allowing a more general type of composition. The expectation was that this algebra would yield some formulations and proofs of new theorems, which would automatically lead to new methods of calculation.

This in the end has proved right, with a lot of people joining in the project. For a long time,

[†]For more information, see https://groupoids.org.uk/hdaweb2.html.

though, for example five years, all that could be said was that it was possible to draw pictures which suggested that the ideas would have to work. The problem was a lack of framework to express the algebra corresponding to the pictures, to the geometry. This framework was built up gradually, and it became ever more amazing to see how natural and fitting a way it was, once the ideas were thought about in the "correct" manner. Thus, as suggested by Wigner in the quotation given earlier, the aesthetic criteria for a proper theory were nicely satisfied, and the theory became better than the vision which had prompted it.

It has to be said that, paradoxically, the secret of success in research is the successful management of failure. For if you never fail, then it is likely that the tasks you have set yourself are simply too easy. Interesting research must have an element of risk. You need strategies for dealing with situations when things go wrong: the problem may have proved too hard, or too easy. What comes next? The analysis of the reasons for failure, and the comparison of these reasons for failure with the reasons for wanting to do this problem in the first place, becomes instructive for future work.

7.5. *Method 5: Writing, rewriting and talking.* A basic method is writing, copying, and talking about and of mathematics. One of us recalls trying five times to write a particular paper; the first four times it ground into the sand, but at the fifth time it kind of wrote itself. Writing and then, after an interval, examining what you have written critically, is essential for getting a good exposition. Do not be afraid of copying for your own use. After several copies, your own original views may begin to emerge. Talking to others, communicating to colleagues and students, is very much part of the progress of mathematics.

8. *What is good mathematics?*

We would not like to attempt to give any final answer to this, but all of us should try and formulate some of the aspects that we are looking for. Indeed, both of us, as editors of journals, have to make judgements on this question on a daily basis. For a new mathematical paper we ask: Are the results new? How far ahead do they go of the current literature? Is the paper clear and well written? Is there a clear familiarity by the author(s) with current work in the field, and the relation of his or her results to the field? How surprising are the results? How elegant are the methods? Are there any new methods introduced?

Some of what we call the best mathematics is that which introduces new ideas and concepts which make the previously difficult easy. This contradicts an impression you may have that mathematics is meant to be hard, and is good for you partly for that reason, like a cold bath. To the contrary, good mathematics can, perhaps should, be easy. It is just that often we do not know how to do this. The combination of apparently simple arguments with a surprising conclusion, perhaps with a surprising twist, is what we like best of all.

What is worrying is that many young mathematicians go through their education without the notion of "good mathematics" even being debated. Yet for any human activity, there is always the question of its value, both for society and personally. There is an argument that the teaching of a subject should reflect something of the values of the professionals in it. For example, for a professional, it is not enough just to produce an answer, but it is important also to produce if possible a satisfying explanation.

We would, thus, argue for the advantages of introducing pupils and students to the notion of good exposition, and even to ask them to compete, not in problem solving, but in producing expositions and exhibitions of mathematical principles and applications. We, ourselves, have found the work on producing a mathematical exhibition enormously instructive [**1**, **3**].

9. *Conclusion*

There is a view that there is no more basic mathematics to be found. This view is comparable to the view of those who have said that physics was ended, the basic problems having been solved.

We feel to the contrary that mathematics is undergoing a revolution, a quiet one, but a revolution none the less. This is occurring on two fronts.

There is first the computational revolution. For computation with numbers, or for graphics presentation, this revolution is well known. Less well known publicly is the computer software which can manipulate symbols and axioms, and other software which can carry out automated reasoning. In principle, these should give mathematicians power to calculate and reason a millionfold more than they can at present, and to deal with the complexities of systems thought previously to be intractable. The prospective effect of these on the teaching of mathematics has yet to be properly understood and assessed, although a lot of work is in progress. The effect on research has already been considerable and is likely to grow in its influence.

A more subtle revolution is the conceptual one. The emphasis on mathematics as the study of structures is finding its mathematisation in category theory, the mathematical and algebraic study of structures. Category theory has revealed new approaches to the basic concepts of mathematics, such as in logic and set theory, and indeed has made respectable the idea that the practice of mathematics needs not one foundation, as traditionally sought, but alternative environments, and a framework for their comparison. These ideas are also important for the progress of computer science, as for example in showing new approaches to data structures. One of the pleasures of mathematics is the way it operates on various levels, which then interact. So the algebraic study of mathematical structures has itself led to new mathematical structures. Some of these new structures have then had notable applications in mathematics and in physics.

Nevertheless, there are still many current dangers for mathematics. There is a general lack of appreciation of what mathematicians have accomplished, and the importance of mathematics. Some of this has come about through mathematicians themselves failing to define and explain their subject in a global sense to their students, to the public, and to government and industry. It is possible for a student to get a good degree in mathematics without any awareness that research is going on in the subject.

Another danger is the growing reliance on computers as a black box to give the answer, without understanding of the processes involved, or of the concepts which are intended to be manipulated. So both the scope and the limitations of the computer fail to be understood, the mathematical basis is neglected and perhaps fails to be developed, and the computer may be used in ways which are inappropriate, or simply limited by the software design. It is said that some engineering firms are dispensing with their mathematical research departments in favour of engineers manipulating software packages. Will this ensure the safety or reliability of the product? Are there computer systems which allow the expression and so use of the most advanced mathematical concepts?

If these dangers are to be averted, then an increased understanding and appreciation of the questions with which we started are essential.

There may be ways of speeding up the process of transfer from the conceptual foresight of the mathematician to the realisation in a scientific or technological application. To find them, we need in society a real understanding of the work of mathematicians, and of the way mathematics has played a role in the society in which we live. It is our responsibility to the subject we love to find ways of developing this understanding.

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[†]For more information see articles on http://www.groupoids.org.uk/publar.html. A question raised by students of "Maths in Context" was: "If you think "Maths in Context" is important, why is it only an optional course in the final year?' A relevant point is that many scientific subjects do give their first year students some indication of exciting current research topics, while teachers in social sciences are shocked that courses in mathematics have nothing on research methodology.

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[†] For further information on Grothendieck, see also http://inference-review.com/article/a-country-known-only-by-name.