The De Morgan Journal 2 no. 1 (2012), 41–67. ISSN 2049-6559

WHAT SHOULD BE THE *CONTEXT* OF AN ADEQUATE SPECIALIST UNDERGRADUATE EDUCATION IN MATHEMATICS?

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Introduction

The main focus of arguments on undergraduate degrees in mathematics is on *content*. However an old school debating society tag is: "Text without context is merely pretext." We argue here that much of the implication of this remark holds in mathematics teaching.

The "context" of the training consists not only of the place it is given, and the background of the students, but also: the relation of the course to the rest of mathematics; what constitutes "good mathematics"; the possible future employment of the students; the way in which mathematics is used in society; the intrinsic value of the subject; how it has progressed over the ages; and so on.

The point we are making is that the study of aspects of this context should be a clear part of the training. This reflects the fact that in any human activity, we need to know the background. If we decide to go on holiday, we don't rush to the station to buy a ticket: we consider what kind of holiday we want; what we can afford; what we can cope with; how to travel; and so on.

The argument for context is not a new argument. The *Preface* to the first edition of Tobias Dantzig's famous book [**9**], published in 1930, states:

> This is a book on mathematics: it deals with symbol and form and with the ideas which are back of the symbol or the form.
>
> The author holds that our school curricula, by stripping mathematics of its cultural content and leaving a bare skeleton of technicalities, have repelled many a fine mind. It is the aim of this book to re-

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2000 *Mathematics Subject Classification* 97C50 (primary), 97D20 (secondary).
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> store this cultural content and present the evolution of number as the profoundly human story it is.

It would be wrong to describe all current undergraduate courses as "a bare skeleton of technicalities". Courses in the history or philosophy of mathematics are common. Nonetheless, the flesh of history, purpose and wider relations is rarely an assessed part of the course.

Our argument for context is related to the notion of "Popularisation of Mathematics": we suggest with [**4**] that we should also popularise mathematics to students, so that they can act as ambassadors for our subject!

It would be interesting to do a survey of graduands in mathematics to find out how their views of mathematics have changed through their degree course. As some small straws in the wind, a woman graduand wrote to one us of to say that:

> "I and my friends ended up with 2:2 class degrees, and were scarred by the difficulty and inaccessibility of the courses. Yet we have all gone on to make reasonable careers."

Another wrote:

> "I loved every minute of my time in a Mathematics degree at Bangor."

A Bulgarian researcher in pure mathematics who obtained an undergraduate scholarship to a top Eastern USA Mathematics course was asked what it was like; he replied: "Three years of hell!".

In considering what should be the aims, or what are sensible aims, for mathematics degrees we have also to consider the fate of the bottom 40%; those who have chosen a mathematics degree clearly have some liking and interest in the subject, and some of them have chosen to compete for what are esteemed as the 'top' universities and courses. It is bad for the subject if some of those who achieve entry to such courses find them hard and boring.

One major problem is that of *assessment*; one research student at Bangor who obtained a very good degree at a major University said that he trained himself to write neat answers to examination questions. The problem was that research involves, among other things, writing expansive developments of areas, and comparing and contrasting treatments in the literature.

One of us (RB) felt that he developed confidence and a long term programme in research entirely through writing a text on topology, and finding as he worked on the text what seemed to him inadequacies and anomalies in the then current treatments; correcting these anomalies opened up a whole new research area. Littlewood's famous book [**13**] remarks that the Cambridge Tripos did not prepare him for research, though it did leave a residual talent, namely that of being able to set examination questions with a silly touch of distinction!

Only a small proportion of mathematics graduates become research mathematicians, but many of the desirable skills of a researcher are also desirable, if not essential, for many of those graduates who find other careers. We should therefore reflect on what some of those skills are, and hence what the 'training' in such skills entails. That then suggests an evaluation or appraisal of current course content and assessment to see if they enhance or impede the acquisition of such skills. In this article, we will not give 'answers' as that does not really fit the bill: we will pose questions and 'reflect' on partial answers as the reflection process is an important part of the 'context' of the task.

Another important point is: what is the impression of mathematics that graduates gain from their studies?

1. *Specialist or professional training?*

"Specialist training" might mean for "future researcher in Mathematics or Mathematics teacher". We would like to replace the term 'specialist Mathematician' by 'professional Mathematician', since this focuses on the well understood idea of 'professionalism'. This replacement is related to our emphasis on context, and to a contrast between the idea of 'vocational education' and 'education for a profession'.

Further, in discussing content of a degree course, arguments as to whether or not, say, Galois theory, or fluid mechanics, are essential aspects of the course may be futile, unless the context is clearly defined. How many professional Mathematicians use Galois theory, how many Pure Mathematicians use Fluid Mechanics? How many Mathematics Teachers use either? The reader may object – but what about Calculus, Linear Algebra, Rigid Body Mechanics, etc., the *really basic* material. Of course, there probably is a 'core' of material without which the student will be unable to understand what is going on in the subject at the present time, and so unable to operate as a specialist Mathematician. But it is not our purpose here to argue for the inclusion or exclusion of certain items in such a 'core'. Instead, we wish to ask: what is the core "for"? why is it the core? how is it to interact with other material? by what criteria should we gauge if something should, or should not, be in the core, and to some extent, how should core material be taught and assessed? These questions cannot be answered sensibly without considering the *context* of the mathematics course.

During a professional career the requirements of the job are likely to change considerably, and so the importance aspects are ability to learn and evaluate. This requires some understanding of methodology, see [**8**].

We have heard it argued that a discussion of methodology in mathematics is either irrelevant or impossible. Against this, we would argue that any human activity can profit by an analysis of aims and methods; that such discussion is a part of professionalism; that researchers in the Social Sciences are often shocked to learn that undergraduate courses in mathematics usually contain no discussion of research methodology; that it is common for workers in the arts to give illuminating comments on their aims in their work; and that in the physical or biological sciences, it is expected that students will obtain from their course some idea of what is going on in research in their subjects.

One of the aspects of undergraduate education for Mathematicians is the wide range of employment of graduates. The main first employment in the UK has been in the Financial area (23%) and Management Services (15%), with only 4% going directly into Science Engineering and support. Teacher training accounts for 6%, and postgraduate courses and training 12%. Many of the latter will become research mathematicians or statisticians in industry, Government, or commerce. Thus the majority employment in the U.K. will not be in a school, institute of higher education, or research institute, and indeed the majority may not use any of the advanced mathematics that has been studied in their course.

On the other hand, they will need some of the desirable skills of 'professional mathematicians', so what are they?

The work of a 'future researcher in Mathematics' has developed in new ways, as other professions examine and define their roles, and the consequent shape of professional education. Twenty years ago, a document prepared for the London Mathematical Society, the Royal Statistical Society and the Institute of Mathematics and its Applications [**17**], states that a report on reducing student overload in first degree courses in engineering in the UK, included a suggestion to "Teach only the mathematics ... applicable to their chosen kind of engineering degree" and a proposal to "Reduce analytical theory". This contrasts with the views of a leading company reported in [**17**] that "Perhaps the most significant change (in the coming years) will be the 'numeracy' required of any scientist, which will put significant pressure on academic course time to incorporate sufficient computational and mathematical skills into general scientists." The mathematics report [**17**] continues: "Where mathematics is not covered by engineering degrees, it will need to be supplied by mathematicians—*but by mathematicians who know to communicate with technologists* (our italics). In fact, there is already a need in industry for well-trained mathematicians with a sufficiently deep and wide range of knowledge to enable them to develop the applications of mathematics to major technological problems."

There are numerous mathematicians working in industry and commerce, in mathematical research areas ranging from Fluid Mechanics, Operational Research, Cryptography, Signal Processing, Computer Science, and applying Numerical Analysis, Optimization, Stochastic Processes, Number Theory, Algebra and Category Theory through to Mathematical Logic. A researcher in Mathematical Logic working in Computer Science on concurrency and parallel processing may need to know aspects of Optimisation Theory, Probability Theory, Stochastic Processes, Quantum Mechanics, or Statistical Mechanics for applications to Neural Network computing. How on earth can one decide on the content of a 'core' to enable such mathematicians to work effectively? This suggests that the 'content' of an undergraduate course is not the key issue. A graduand of Bangor who moved after his doctoral work in topology to work for an engineering firm wrote:

> "We can get all the computer scientists we want, but a mathematician who can apply a range of appropriate techniques to the problem at hand is worth his weight in gold."

One aspect of such work is that these mathematicians will need to "know how to communicate" with others, who will usually not be specialist mathematicians. This implies that they must to some extent explain, and "teach" the Mathematics, but they must also be able to get into the mindset of their "clients" to see what mathematics might be available, or need to be developed or adapted, to express the ideas the engineers or biologists are using. Part of the professionalism, therefore, of any mathematician should be the ability to *communicate* Mathematics and so it seems reasonable to suggest that training in such communication should be part of the *education* of a mathematician, even though the majority will not go into the teaching profession. In fact, our experience showed that enabling this aspect of their professionalism, actually helps their technical expertise in other aspects of their degree course including of the 'core'.

2. *What is a Mathematics Degree for?*

In deciding the relation between 'content' and 'context' in any 'training' programme, the first question to ask of that training is its purpose.

If it is to be a training of a 'Professional Mathematician', what does that entail? What attitudes and skills are involved? Should the degree course encourage the learning of skills and techniques that will be transferable to other situations? What are such skills and techniques? Can they reasonably fit into a degree course in Mathematics? If so, how?

Whatever our answers may be, we certainly need to preserve the 'quality' of the graduate mathematician. Thus those who teach such a course should be asked whether they feel their courses produce graduands with the qualities they, as teachers, desire.

3. *Quality*

As professionals involved in the "production" of some "end-product", how can we gauge the quality of that 'end product': the mathematics graduate? In the U.K., we are constantly being reminded of the need for "quality control" by the government, with the professed desire to make certain public money is being well spent. What is this quality that we should control? What does "well spent" mean?

In industry, quality control is based on the adage: *Quality is decided by the demands of the customer, not the declaration of the producer.*

Is this applicable or adaptable to the context of Mathematics courses? If so, who is the customer? Is it the student, the potential employer, "Society", whatever that may mean?

The report [**17**] states that: "The need for mathematics and mathematicians is an expanding and all-pervasive aspect of a modern science based economy." This suggests that one of the aims of the courses should be to ensure a competence and understanding of mathematics to see this need and to provide for it.

If the 'customer' is only the student, there are problems: students are naturally fairly immature when they enter higher education; they have nothing with which to compare what they are "getting"; no criteria are available to them. Their views should be sought, but must not be the only source of "feedback". If we are professionals, the responsibility for quality must surely rest with us in the last resort. On the other hand, to what extent should students be expected to enjoy the course? to understand what it is about? to feel they are expressing themselves? to be convinced it is of value to them? Today's student is tomorrow's parent, teacher, employer.

If the customer's role is played by the employer, can we find out what they think of our 'finished' product? No recent survey of employers' views has been made within the U.K., the latest reasonably thorough one would seem to be the McClone Report of 1974, [**14**]. As summarised there, the employers' view of the strengths of the mathematics graduate include:

(i) knowledge of mathematical technique;
(ii) ingenuity;
(iii) capacity to seek out further knowledge;
(iv) ability in problem solution.

On the negative side, the mathematics graduate is:

(i) poor at formulating problems,
(ii) poor at planning work,
iii) poor at making a critical evaluation of completed work,

and in any case it did not matter too much what they did since

(iv) they had little or no idea of how to communicate it to others.

Has much changed since 1974? We leave the reader to ask themselves. There have been changes to the way things are taught, to the content and the means used. The important thing to ask is which of these changes have been a success from the point of view of the criticisms in that 1974 report.

4. *The skills of a professional mathematician*

The McLone report suggests as a summary of necessary professional skills (at least as viewed by potential employers in industry and commerce of that time):

(a) Planning the work (Aims of work, available methods of attack, time constraints, etc.);
(b) Formulating problems and solving them;
(c) Knowledge of mathematical techniques and how to apply them to a variety of situations;
(d) Knowledge of the literature (and not only that part encapsulated in the student's lecture notes, but how to select which books, journals, etc. to read - and how to read them);
(e) Making a critical evaluation of work done, both at interim stages and after completion.
(f) Communicating the result of a piece of work to others at various levels of complexity, not just to fellow mathematicians;
(g) Working in a team and working independently;
(h) Ability to seek out and master necessary new material, i.e. to learn;
(i) The ability to produce finished work of a high standard of presentation.

It may validly be objected that, apart from (c), these are the general skills of *any* professional. Many of these skills and 'techniques' desired by employers are not subject specific, nor is their lack restricted to Mathematics graduates. A mathematician working for a software designer may not need much calculus, but will need the above skills. A mathematician working for an oil company will probably not need the theory of Boolean algebras, but will need these skills; and so on. They are not subject specific, but that implies that they are also relevant to the training of the undergraduate mathematician who will not end up as a 'professional mathematician'. Perhaps this suggests that they are some sort of "bottom line" applicable across many subjects. If that is the case, then these skills are of general *vocational* relevance. Words such as 'vocational', 'training', 'formation' and so on, are often put in opposition to 'academic' and 'education', sometimes to the extent that 'academic' is often used almost as an insult in political debate, i.e. "academic" = "irrelevant". However, although, these skills *are* vocational, they are also *academic*, in the sense that they are necessary skills for the research mathematician and for the mathematics teacher.

If students already had these skills, they would be so much more easy to teach. They might learn to learn!

Are there any subject specific skills that should overlay these general skills? It is easy to suggest a few:

(j) Understanding of the use of mathematics in the modelling of aspects of the "real world". (Design, analysis and limitation of models).
(k) Appreciation of the conceptual and descriptive power of mathematics.
(l) Appreciation of the notion of mathematical validity, that is to read, understand and write proofs.
(m) Skills in avoiding 'slips' in calculation, etc.

This list is by no means exhaustive. It suffices to listen in to the 'moans and groans' overheard in many common rooms in mathematics departments to make a fine and rich collection of skills that are apparently lacking in our students!

We would identify various currents throughout these skills. Viewed for their implications for course structure, in part they correspond to:

(i) an appreciation of the interactions between the various parts of Mathematics and between Mathematics and the "rest of the World";
(ii) an overview of Mathematics and of its system of values;
(iii) a much greater overall mastery of content.

The best undergraduates show many of these skills without our intervention. Very many more show some of these skills. Can we aid the students to develop their skills *through* Mathematics? Can we choose the content, and our means of delivering that content, in such a way as to encourage these skills?

The impression of Mathematics usually given to students is that it has a tree like structure, which can be learned only successively, understanding one part before going on to the next. Further, Mathematics is presumed to be fully formulated, tidy and neat, not necessitating any leap of the imagination; rigour is the main characteristic, not intuition, or ideas; it is a difficult subject, with a high risk of failure. A global view of the purpose of the subject is not available, and popularisation is almost impossible, or so it is implied. And it is not possible to give first year students, or even final year students, any idea of current trends in the subject.

The real story of Mathematics is usually hidden. Students can do courses:

- in group theory without understanding the wide applications of the notion of symmetry;
- in the calculus without understanding the centrality for applications of concepts of motion, continuity and rate of change;
- in topology or functional analysis without understanding the richness and wide importance of the mathematical notion of space.

Anyone working in mathematical research knows both the truth and the superficiality of the tree-like view of Mathematics. To decide on the structure of courses and assessment on the basis of this view is to base the pedagogic structure on poor foundations. The problem would seem to be that to analyse *dispassionately* the conceptual structure more fully is hard, and assigning the "correct" weights to different aspects is especially so. What should be the balance between *knowing* and *doing*, between *theory* and *technique*, between understanding the *mathematical process in general*, and *accurately performing a particular mathematical technique under examination conditions*? How does one build on the basics? Can we build an understanding of modelling, say, when the students cannot differentiate, or cannot solve simple differential equations?

These questions are perhaps badly posed. Perhaps we should be asking more constructively: what are we doing that might give the wrong messages to students? We ask them to jump through hoops, and hope they will do this not just "accurately", but with enthusiasm and love for the task. Unfortunately, they continually fail to do so as we wish. There is clear evidence that a direct *technical* education fails.

It is a truism in psychology that if a high proportion of a population behave in a certain way in a given situation, then this behaviour can be regarded as a "reasonable" reaction to the "controls" inherent in the situation. If, for example, students do not use the library in a mathematics course, it can be asked whether the students perceive that use of the library is not particularly advantageous to the successful conduct of the course. It may be, for example, that all they need to know is at some time or other written on the black board. So we have to be clear, and make clear, as to what is expected of students, what we require of them, and design the assessment to reinforce this. The assessment is the main control mechanism. Changing the style of assessment is very likely to have an effect on the end behaviour of the students.

There is also a contrast between technique and value, between craft and art. The training of say a concert pianist, requires the practice of scales and arpeggios, getting into the fingers the necessary technique. But students of the piano are also required to understand the music, to show musicality. Should we be considering some form of "mathematicality" of our mathematics students? and to present both technique and "mathematicality" to the students?

To take another metaphor, in training a *chef*, one does not present the trainee *chef* just with finished meals, nor just with the task of peeling the potatoes. A trained *chef* is required to design and produce the finished product, the meals, to a high standard. In [**2**], the point is made that a valuable course in carpentry is one in which the student uses particular skills to make a finished product, on which the assessment could be based. One reader of [**2**] remarked that he had taken a carpentry course exactly as that satirised!

Courses in mathematics rarely state their overall aims. In contrast, one course in design stated the following:

> "The aims of this course in design are:
> (a) to teach students the principles of good design;
> (b) to encourage independence and creativity in the student;
> (c) to give students a range of practical skills which will enable them to apply the principles of good design in an employment situation."

In discussions with colleagues as to how mathematics courses score on analogous aims, the suggestion is usually between 0 and 2 out of 10. Why is this? Are these aims unreasonable in themselves? Are they impossible of attainment?

Some would retort: there is no agreement as to what constitutes good mathematics, so how can it be taught? ... but that is the whole point. It does not mean that there should be no discussion of that question. Quite the contrary. Lack of discussion leads to a situation where a student can get a good degree mathematics and not be able to formulate an opinion on: "What constitutes good mathematics?". There is also a danger that a reply might be: "Good mathematics is what the top mathematicians do."— which would be typical for its naivety.

The trainee mathematician should be able to 'peel the potatoes'— but is that enough? Do we discourage the development of the 'mathematicality' that both lecturer and student desire to some extent, by a diet of 'spud bashing' and completed 'cordon-bleu' mathematics? This does not encourage an overall view of mathematics, nor does it allow the critical evaluation of 'work done' necessary for the development of 'mathematicality', and if the students do not write mathematics even for themselves, they can scarcely communicate it to others.

Can we encourage investigation of a problem by the students before the technical aspects are fully explained? Certainly not all subjects within Mathematics lend themselves to such an approach, but the more algorithmic parts can provide experience of comparison of partial solutions with full ones, of evaluation of different approaches, and a deeper appreciation of what makes the algorithm tick.

If we decide on a common 'core' of mathematics needed by the 'professional mathematician' and thus to be included in all acceptable training programs for 'specialist mathematicians', where should we stop? In reality, mathematics is *not* linear, nor even of a tree like structure. There are innumerable cross links, and even more important, one may need to see the end of a subject before one can really understand its logical beginnings.

A theoretical physicist may need to *learn* knot theory, a low dimensional topologist may need to *learn* new material without it being served on a plate—does the current diet of lectures prepare students for this?

Similar problems arise for the industrial mathematician. Learning is often most effective when there is a clear "need to know" factor. This allows for planning, decisions, and evaluation as to how much to know and how best to learn what is required. It is this factor which is usually lacking, since the actual work of a mathematician is not something of which students are well aware. Indeed, such an analysis may be too difficult for many staff. We come back to the astonishing fact that the *methodology* of mathematics is one of the least discussed subjects of all!

Although we probably need to preserve much of the majority of the basic content, as there is a professional consensus as to its worth as an accepted body of knowledge and techniques, can we teach it in ways that will help develop these other skills? Can we teach it so that students can use it to build their own 'mathematicality' to the limits of their own ability?

5. *Lectures and tutorials*

It is commonly held that the lecture, as a means of transferring words from the notes of the lecturer to the notes of the student without passing through the mind of either, is quite efficient. The lecture is, however, more time consuming than photocopying and where photocopying is available many students avail themselves of this alternative! Of course, a good lecturer not only transfers material, but inspires interest and enthusiasm in the subject, but that form of interaction may be distinct from the transfer of material. One hears of lecturers whose technique goes against all the "rules" of good lecturing but whose personality fires the enthusiasm of the students who go away and read text books to construct their "lecture notes" from the rough skeleton gleaned from the lectures. That same lecturer talking on another subject or to another class may not have the same effect. The lectures worked because they encouraged the students to take the initiative and develop learning skills. A student who did not take the initiative would be lost.

Lectures are far from perfect, but perhaps the main problem is that they are seen by both students and lecturers as being where most of the work is done. If they were seen as merely the starting point of the students' learning process, which is really as they should be seen, then it might be easier to use lectures for conveying enthusiasm, and encouraging the desired knowledge, techniques and skills. Lectures then might provide an overview of the subject and an insight into the problems the student will face in individual study. This viewpoint would suggest that, for instance, online notes should be skeletal and the presentation of the material within the actual face-to-face lecture may need to be revised.

Many lecture courses already do provide an overview, showing the interaction with other branches of Mathematics or applications to other areas. Students tend to react by saying "Old so-and-so is waffling-on again; as he cannot ask examination questions on that stuff, I will go back to sleep". Worse if students realise that the lectures consist only of "context" rather than "content", they will prefer to spend their time in the coffee bar! Perhaps that would be an improvement!

Here we are getting to a key point. How does one test the skills that have been outlined earlier? If they are not assessed, they will not be taken seriously. Lecture based courses, as the only item on the teaching menu, would seem to encourage the wrong habits and are linked with assessment only of content. Again content is important, techniques are important but so is what we have called "context". If "context" enhances learning of "content", as we suspect, how can one teach it, how can one, assess it so that it is taken seriously by the students?

One of the aspects of 'context' is an appreciation of strategies for solution and proof. 'Telling' students how is usually not enough. Lectures can, however, be very useful for showing how a proof is *found*. Often students' grasp of basic logic is flawed or wanting, but In a lecture one can, for instance, present a proof that for particular sets A, B we have $A \subseteq B$ by presenting (or asking for!) the first line, and the last line of the proof, and then showing how the middle is filled in, working from both ends.

Again, much teaching must start from observation, and then trying to move the learner from level ℓ to level $\ell + \varepsilon$. One of us, RB, liked to conduct a tutorial, particularly a small group tutorial, without writing anything on the board, but encouraging the tutees in a non stressful way to write themselves, for example simply writing out the question they could not do, as a start, and then continuing even with some dictation as necessary. It was noticed that many students came with few strategies! As an analogy, a piano teacher must listen to the pupil, not just rattle through the piece in his or her own way.

6. *Other teaching techniques*

There is widespread awareness of such problems in the aims and methodology of teaching mathematics. Many innovative methods have been tried with varying degrees of success. We will briefly describe and comment on some of those we know, before passing on to a course that we have developed at Bangor which we will describe in more depth and detail.

Schoenfeld [**16**] states:

> The activities in our mathematics classrooms can and must reflect and foster the understandings that we want the students to develop with and about mathematics.

If we want the students to start developing the skills we listed above, we must make sure that class room activities reflect this. If the activities are too costly in manpower or in capital expenditure, such as many computer based courses, then many institutions will find it difficult to implement those activities in their classrooms. We thus concentrate on "low cost solutions" and also on pragmatic criteria for their implementation - you cannot start from scratch, you always start from the presentation in your establishment.

6.1. *Problem Solving*

Solving problems was one of the mathematics graduates "strengths" in McClone [**14**]. Can we use it to help with the other skills?

Schoenfeld [**16**] described a problem solving course he ran. The aims of this course included helping the students to develop their mathematical judgment, to "understand, justify and communicate mathematical ideas". (We might add that by using various different styles of structure, Schoenfeld's course also allows students to work in teams. He did this in part by allowing the students to learn by making a tactical withdrawal from the class.)

He also tried to get students to see Mathematics as a human activity with a set of criteria for validity—but *not one* imposed from outside, *not that* decided only by the lecturer. The students evaluated their own efforts by defending their views and attacking contrary ones, until consensus as to the valid argument was reached. To do this, Schoenfeld created an artificial environment to provide students with "a genuine experience of *real* mathematics". He concludes:

> "By that standard, standard mathematics instruction is wholly artificial."

We would add that a problem solving course as described by Schoenfeld has the advantage that it requires no large computer laboratory, no input from local industry, no expensive equipment. It is therefore within the reach of many institutions unlike some other innovations which are costly. It is however labour intensive, requiring a lot of dedicated input and preparation by the staff members involved. It may also require training of staff members, as it requires new teaching techniques. It is not always clear where such training can be obtained.

6.2. *Error-less learning*

A technique widely used by psychologists and trainers is *error-less learning*. This falls into two types. One is where large hints, props, and supports to a specific course of action are given, and the action is rewarded as a symbol of success. Then the various props are gradually withdrawn. The other type uses *reverse chaining*: the easiest way to see to this is to think of encouraging a child to put on a vest. You do not throw him or her a vest and say put it on; instead, you put it almost on, and then ask the child to do the final action. Subsequently, you gradually put the vest less and less fully on, till the whole action can be done.

One way of using the last technique in university mathematics is to write out a formal proof and then erase bits of it. The student has to fill in the bits, using clues from the rest of the proof. This has some analogies with the practice of a professional mathematician, who may have an idea and outline for a proof, but needs to work on details. The student also gets an idea of the structure of a proof. Such an exercise is also very easy to mark!

The general feeling about error-less learning is that it works like a dream!

In either method, the fact long verified by psychologists is used, that *we learn from success*. We can also learn to accommodate failure if that is gradually introduced, and strategies are available for dealing with failure.

6.3. *Modelling*

"Problem formulation" was diagnosed as a weakness of mathematics graduates, and indeed there is little written about it.

Yet one form of it, modelling, is an essential skill for graduates (not only for the "applied" mathematician, as "pure" mathematicians model geometry by algebra etc., all the time). Two areas in which modelling is relatively easy to set up are in elementary mechanics and in operational research.

In the UK, mechanics has disappeared from the options offered by many schools, being replaced by probability and statistics. As a school subject, Physics has also been on the decline for various reasons. Students arrive at university having had little opportunity to reflect on the interaction of mathematics with "mechanical" reality.

As a result, students tend to find mechanics hard and even unintuitive or too abstract. This has led several institutions in the U.K. to experiment with practical modelling sessions, in part reintroducing the "physics laboratory" session that would previously have been available in schools. The equipment needed has been developed jointly by a group of institutions and is designed to be relatively cheap to buy, mostly being assembled from "toys".

The idea of a laboratory session is old, but that does not mean it is "bad". It can provide opportunities for applying knowledge to the formulation of a model, problems (at what angle of slope, will the car fall off the circular track?), evaluation of results so far (my model predicted that at this angle the car would loop the loop—it didn't manage it; where was my model inadequate? how can I improve it?), working in teams and communicating the results in a report. Emphasis may be placed on only some of these skills but again the student is *doing* mathematics.

Similarly, in operational research, several universities have experimented with bringing in *real* problems for the student(s) to handle. This can be as an individual project or as a group project on traffic flow, how a local timber firm can best cut or stack its wood to minimise waste or wasted space. The problem may be small, but real. This also provides opportunities for communication, since if a local firm has provided the problem, it should receive a readable report. Here, "readable" means "readable to the non-mathematical management of the firm". Group work is useful here both logistically, realistically and for the advantage to the students. (Assessment is no problem in practice as there are well tried methods of evaluating group work.) In practice, students do not know enough, say, linear programming to solve a real problem—so they have to take the initiative and find the additional theory. This does require a suitable library. They have to hand their reports in *on time*—so time constraints and planning their work are essential. Here, to be honest, there is a problem. Students do need help on these aspects and many lecturers are not too good at meeting deadlines, planning their work, etc., so once again the lecturers may need to develop the skills first! There are also problems if then the local area does not have much industry as a supply of "real life" problems is hard to get. Here cooperation between institutions is called for, as a real problem has aspects that preformulated ones do not.

At a less ambitious level, one of the authors (TP) used the method of introducing the students in an O.R. course to a problem type, a week or so before the topic would be handled in the lectures. The students would discuss the problem in a tutorial / problem class situation and some guidance was given to direct their directions of attack on the problem. It seemed that this helped students see in what way the theory refined their efforts at a solution and hopefully helped them in understanding the problem and the modelling processes being used. The students' reactions to this were favourable in the main and they seemed to appreciate the challenge of attacking a problem without the certainty of finding a solution, (but at no risk to their assessment grade and little to their self estime!).

Finally there is a difficult question: how can this sort of activity be balanced against content? If you introduce such a course, some explicit content will have to go or the students will be overloaded. If however, the students, by means of such a course, learn to seek out information, it may be that their efficiency in learning the content that is presented to them will increase. The content of a course, however good, is useless unless it is appropriate to the students, and they can learn it. Merely putting it in front of the students is not enough!

Do students benefit from such modelling courses? This is difficult to answer. We know of no studies which evaluate such courses from the point of view of improving achievement in assessed work. There are, however, reports that 'students do enjoy and participate fully in such courses'. This may merely be an indication that the diet of lectures is boring by itself and such a course acts as a chance to show creativity rather than being just a "sponge".

There is also the general question of the place of the term "analogy" in mathematics courses, which is central to the modelling situation, but has much wider applicablility than in those areas in which we normally think of modelling is occurring. One of us (RB) gave a presentation on knots to school children and teachers in the 1980s, and explained about prime knots. He also said that analogies are not between things, but between the relations between things. Thus knots are not analogous to numbers, but an addition of knots can be seen to have properties analogous to those of multiplication of numbers, leading to questions such as "Are there infinitely many prime knots?". After the lecture, a teacher came up to RB and said: "That was the first time in my mathematical career that anyone had used the word analogy in relation to mathematics!". Yet the wider possibility of analogy is behind the power of abstraction! Often students seem naturally to use some form of analogy in their thought patterns, but they question if it is a legitimate tool to use in solving problems. We would suggest that it is thus a powerful tool for teaching as analogy frees up thought for the 'that reminds me of' moment, but also has its limitations and that is where ideas of proof and rigour come naturally to the fore.

6.4. *Precision questions*

The usual way the 'set work' in a mathematics course is handled approximates to the following: it is set; it is marked (often with indications given as to how it might be improved); it is handed back; the student glances at it, and files it away. If you are lucky, the student may look at it again whilst revising to check the method of solution.

The aims of *precision questions* are to help student to reflect on what is required for a full and precise answer and so to develop the awareness of presentation and communication. At Bangor, we have tried this with exercises in some First Year Classes.

The first draft solutions were handed in to tutors in the eighth week of the ten week first term. They were corrected with suggestions for what was lacking. Rewritten, they were handed in a second time in the eighth week of the second term. Although few marks were awarded for these, most students seem to have made a concerted attempt to improve their first drafts as suggested.

In view of the current ubiquity of computers, students can easily learn to use LaTeX, and to see its advantages for presentations—the challenge of producing that quality of output often interested them. The second version often needed still more work, but for many the effort of writing the improved version seems to have led to a great increase in the quality of the solution. Such type of quite simple adaptation of existing practices may aid the student in communication, in evaluation of their own work and in gaining an appreciation of what is a valid solution to a problem.

The first author has written in a *Preface* to [**3**] of the advantages to research work of writing and rewriting expositions to make the matter clear.

6.5. *Mathematics in context*

This is the type of course that aims to reach the parts that other courses do not reach. We are biased! The immediate apparent aim was to improve the students overview of Mathematics: how Mathematics is formed, how it is applied, how is it (and also how might it be), taught throughout the ability and age ranges? Are there current issues in Mathematics? In what sort of mathematical areas is there current research going on? Mathematics is a human activity. It interacts with Society. How is Mathematics funded? Can Mathematics be made more "popular", more accessible to the general public? Mathematicians form a group within Society. How do they interact with each other? What is mathematical validity? What is proof? Are questions of validity social and human matters, or are they internal to mathematics? How can one approach these questions with a reasonable standard of academic rigour?

There are far too many potential topics to cover, so we picked some often in consultation with the students, inviting external speakers where local expertise is lacking. The sessions can range from a lecture of a reasonably standard type, to a problem solving session, a discussion on popularising mathematics to a simulation of a training session for 'quality control' in the steel industry using some statistical methods the students had not met. Where is the established body of knowledge on which such a course can be built? There is none – but that is marvellous. The students are "embryonic" mathematicians. They can hold valid views on many of these questions. The only requirement is that a view should be justified.

A student (at another university) was asked why he wanted to study to be an actuary, even though he had a very good degree: he is reported to have replied that nothing was happening in Mathematics, no research was going on, so continuing would be too boring. One of the first things we ask our students each year was to go to the library and find Mathematical Reviews. (This was a few years ago and now they could be showed the online MathSciNet and let loose on it!) They are told what it is, that it is a potential resource for their work, and that we want them to comment on their reactions to it, and to report on some aspect they note about it for the following week. The reactions of the students varied in detail, but was very interesting and as some of them may read this article, we will not reveal what sort of reaction they had.

Naturally enough mathematical education was one of the themes. Students have often reflected amongst themselves on the nature of their training in mathematics at secondary school, or at university. We tried to get them to bring that reflection into the classroom, to discuss points with us and hopefully thus to obtain insight into their own learning processes and what this learning process says about Mathematics as a human activity. The way this was done involved external speakers who talked about some new project in school mathematics. The speakers had, of course, been briefed as to the aim of the course.

Students are well aware of the unpopularity of mathematics, or the surprise that girls do mathematics. As part of the discussion of this, we arranged a visiting speaker from a Museum of Science and Industry, and we, ourselves, explained the purpose and the problems of design of our *Mathematics and Knots* exhibition (see [**5**, **6**]). But you cannot popularise mathematics without a clear aim and message. Thus popularisation goes hand in hand with a clear view of the nature and context of mathematics.

One point of discussing these matters with students, and indeed a point of the whole course, was to give students understanding and confidence in their subject of study, so that they could, in casual conversation, explain and defend their decision to study mathematics. As an example of success, one student at an interviewing board was asked about the Mathematics in Context course. She reported to us with pleasure that every question she was asked had been discussed at some time or other on the course! She got the job. Another student wrote in his project that writing it had helped him to come to terms with his attitude to mathematics and yet another student wrote in her project that lecturers think that students do not discuss mathematics: but the lecturers do not see students arguing among themselves!

These reactions to the course were to us a surprise and pleasure. Indeed, the students totally surprised us with their independence and initiative. We could never have imagined that any of our students would be able to write with such imagination and clarity on Mathematics and Art, or Mathematics and Music. A course on Mathematics and Society at Liverpool University resulted from discussions with one of us on our course, and in that course also, the organisers, Roger Bowers and Brian Denton, say they were bowled over by the quality of the presentations. Thus it may be that the standard mathematics courses have failed to capitalise on the universal nature of mathematics, the fact we all need to work with the geometric, logical, numerical aspects of the world around us. There is evidence that a return to the source of mathematics, its context and nature, can revitalise students' attitudes to the subject, and place within it, by giving them an orientation, an understanding of their subject, and so help them to cope with its difficulties and technicalities.

A third theme was the interaction of Mathematics with other Sciences and with Industry. Here we had a small problem as we are in a rural area with no large industry near. Our contacts were greatly helped by our colleagues at Liverpool who, as we mentioned, ran a course that was similar in conception but slightly different in structure.

The sessions aimed at putting Mathematics into Context, that is exploring aspects of Mathematics in new lights in an attempt to encourage the building of an overview of the subject. If the students can start to put Mathematics in Context, perhaps they will start putting the other courses they are taking into context as well, creating a more coherent whole.

The practical details of the course were that it was a final year optional course; it was twinned with a History of Mathematics course and was worth one twelfth of the final year marks (that is, equivalent to one half of one 3 hour examination). In 1991–92, 15 students took the course. (There were about 45 students in the year.)

In [16], we described the course eighteen months after it had first run. The ideas were still the same, but certain details of the assessment had been changed. Our aim was to design the assessment so as to aid the development by the students of the "skills of self-evaluation, communication, planning of work and selection of material (i.e. knowledge of the literature)". The students had to prepare a report/essay/project on some subject within the broad framework of "Maths in Context". After an initial planning stage, they would discuss their ideas with one of us. At this stage, it is noticeable that their projects being self selected, did tend to be too ambitious and one of our first tasks was to help them cut the area down to size. The students often needed a slight push on the literature search as it is not a skill they were used to handling. We did provide a list of potential titles for the essays but a student with a definite idea other than from the list was encouraged to choose their subject, if we thought, after discussion with them that it was feasible. It was impossible for us to do more than suggest some initial mode of attack, perhaps one or two titles, or authors. Once started the student was asked to produce a 'pilot study' after one term. This could be a draft chapter of the final report, or a skeleton of the overall project filled out in parts. This 'pilot study' was to test the feasibility of the project and was both, for us, a guarantee that a student was not biting off more than they could chew, and, for the student, a form of check that they were 'on the right lines'. The draft was marked by both of us and the marking was discussed in a detailed interview with one of us. (This was very time consuming.) It seems that already by this stage, students were evaluating their own work and were themselves trimming the task to size, but they still needed help in achieving acceptable standards of presentation and keeping the topic within the general aims of the course.

We also asked students to conclude the project with a self-evaluation of it, of any difficulties in carrying out, and how it might proceed given more time. This was important as unforeseen difficulties might arise, and not be possible to be overcome in the time allocated. Thus an assessment of how the project could have proceeded given more time was not only an important part of a professional approach, but could rescue a difficult project.

There was a tendency with some students to want to describe, say, an application of mathematics, but without attempting to reflect on why mathematics could be applied in that situation, or what was the success of the application. We did not demand a definite answer – there might not be one – but we did not just want page after page of lectures note style material either. One way out that seemed to work quite well in this situation was to adopt the style of a popular article (*Scientific American*, *New Scientist*, etc). This made the communication aspect explicit, and also forced the student to reflect on what the mathematics was saying. At this point, we should say that there also needs to be some advice and checks on the avoidance of plagiarism.

The final project had to be handed in at the start of the third term (week 21 of the teaching year approximately). Various styles of finished project have been successful. Students have aimed for popular articles, and we have also had an exhibition on fractals with supporting documentation, a hand written beautifully illustrated, "coffee table" book on Mathematics and Art, material prepared for a Mathematics Masterclass for 13 year olds, and a feasibility study for the inclusion of an option on Mathematics and Music in the advanced level examinations at the end of secondary school.

A visitor to the School who was shown the context projects from previous years commented that it was noticeable how the projects gave the impression of being the personal statements of the students, made because they had something to say about their subject; our colleagues at Liverpool reported the same phenomenon in their analogous course. This involvement explains why we changed from two medium sized reports, with one on History, to just one report on Context. The students put too much effort into the preparation of their reports for the percentage of their final marks that it would influence. The 'pilot study' had also an important aspect of training, since students have little previous experience of writing essays, and in this course they needed guidance as to what was expected.

One problem that was mentioned by students was: if "Mathematics in Context" was important, why was it an optional course in the final year? For this reason, we introduced a first year course "Ideas in Mathematics" where the lecturer was completely free to organise it at will. For a couple of years it gave a small introduction to some Euclidean Geometry. There were three reasons for this.

(i) This subject is part of our heritage, and students should surely know at least why the angle in a semicircle is a right angle.
(ii) There are some lovely and surprising results.
(iii) It introduces students to the idea of proof, where the proof proves something striking.

A simple example is the following. Let ABC be a triangle in the plane. Choose points D, E, F on BC, CA, AB respectively. Then the circumcircles of the triangles ADE, BEF, CFD meet at a point. There are further developments of this situation. Results such as this may be seen as ends in themselves, rather than have the first year consist of learning results and techniques which will possibly be used at some later stage. For more discussion of one year of this course, see [**1**]. The aim was a first year course which was not just a preparation for things to come, but which should directly touch the imagination, and suggest that mathematics is an area where new ideas, new ways of seeing things, are being found.

As another example in which contact with the wider world was important, we ran at Bangor a second year course in analysis which included a fairly light hearted introduction to the notion of fractal via the metric on compact subset of $\mathbb{R}^2$, without too much proof. One argument for this course was that fractals and chaos are part of a quite general public knowledge, and it is only reasonable that mathematics students should have seen some of the mathematics behind these notions, without necessarily going into the hard features of that area. One small assessment given was: write an essay on the importance of fractals. Students were encouraged to use books, the web, fractal programs, and to copy and paste to make a nice exposition.

Finally, one difficulty mentioned by a visitor from Spain was that you can't teach context to the students until you have taught it to the staff. Brian Griffiths wrote similarly in an email to RB that lecturers have never had lectures of this type. All one can say in response is that it is sad if a cycle of deprivation continues.

7. *Conclusion*

If we are training specialist mathematicians, we need to ask what are the desirable qualities of the 'end product'. The activities used in the 'training' process, and the assessment, should then reflect those qualities. It would be naive to expect that the end result of such a training process is going to be a fully mature and competent mathematician – one does not expect musicians leaving music academy to produce a mature playing style until after several years of practical application of the skills they have acquired – but the present diet of 'content' with no reward for 'context' does not prepare many student for their future 'apprenticeship' as professional mathematicians, neither does it prepare those students who will not continue working with their subject, for many of the tasks that are required in employment.

Several of the desirable skills that we have identified above are able to be encouraged with courses that are quite able to run without huge expenditure of resources. (The detailed adaptation of such ideas to the local educational structure of an institution cannot, of course, be treated here as each case will be different.)

The Mathematics in Context course at Bangor that we described and its 'sister' course at Liverpool may be a case of too little too late, but they have the advantage of allowing the students to develop some "mathematicality" and reflection on the subject, as well as providing an opportunity for other skills to develop.

Our discussion also suggests other questions, such as:

(a) How much interaction should there be between specialist and nonspecialist undergraduates in mathematics?
(b) Should prospective teachers of mathematics at secondary school level take the same courses as prospective researchers?

We do not wish to answer these directly and finally. Rather we would argue that specialist and nonspecialist, teacher and researcher, should be exposed to discussions on the nature, context, history, of mathematics, and that in such a course each can give valuable contributions. All of our students should obtain a clear impression of a sensible professional approach to the subject.

A course for professionals, and this includes both specialist and non specialist, teacher and researcher, should include not only technique, and knowledge, but also a sense of value, an idea of what is "good mathematics", why it may be called good mathematics, and what are the areas of debate in such a judgement. Without context and value, the course becomes dehumanised, and students can become confused and so demoralised. Thus a lecturer might get the impression that students are interested in a degree in mathematics *only* as a route to a good job. Might this be a case of: "They ask for bread and are given stones."?

We do not attempt to provide solutions to the questions we have posed. 'Solutions' might tend to rigidity and strangle the life of an idea. Rather it is the continual (re)examination of Aims and Objectives of degree level courses that we suggest is a necessary means to improve the training of the specialist mathematician.

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