

## MATHEMATICAL INDUCTION

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INDUCTION (Mathematics). The method of induction, in the sense in which the word is used in natural philosophy, is not known in pure mathematics. There certainly are instances in which a general proposition is proved by a collection of the demonstrations of different cases, which may remind the investigator of the inductive process, or the collection of the general from the particular. Such instances however must not be taken as permanent, for it usually happens that a general demonstration is discovered as soon as attention is turned to the subject.

There is however one particular method of proceeding which is extremely common in mathematical reasoning, and to which we propose to give the name of *successive induction*. It has the main character of induction in physics, because it is really the collection of a general truth from a demonstration which implies the examination of every particular case; but it differs from the process of physics inasmuch as each case depends upon one which precedes. Substituting however demonstration for observation, the mathematical process bears an analogy to the experimental one, which, in our opinion, is a sufficient justification of the term ‘successive induction.’ A couple of instances of the method will enable the mathematical reader to recognise a mode of investigation with which he is already familiar.

Example 1.—The sum of any number of successive odd numbers, beginning from unity, is a square number, namely, the square of half the even number which follows the last odd number. Let this proposition be true in any one single instance; that is,  $n$  being some whole number, let 1, 3, 5 up to  $2n+1$  put together give  $(n+1)^2$ . Then the next odd number being  $2n+3$ , the sum of all the odd numbers up to  $2n+3$  will be  $(n+1)^2 + 2n+3$ , or  $n^2 + 4n + 4$ , or  $(n+2)^2$ . But  $n+2$  is the half of the even number next following  $2n+3$ : consequently, if the proposition be true of any one set of odd numbers, it is true of one more. But it is true of the first odd number 1,

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for this is the square of half the even number next following. Consequently, being true of 1, it is true of 1 + 3; being true of 1 + 3, it is true of 1 + 3 + 5; being true of 1 + 3 + 5, it is true of 1 + 3 + 5 + 7, and so on, *ad infinitum*.

Example 2.—The formula  $x^n - a^n$ ,  $n$  being a whole number, is always algebraically divisible by  $x - a$ .

$$\begin{aligned} x^n - a^n &= x^n - a^{n-1}x + a^{n-1}x - a^n \\ &= x(x^{n-1} - a^{n-1}) + a^{n-1}(x - a). \end{aligned}$$

In this last expression the second term  $a^{n-1}(x - a)$  is obviously divisible by  $x - a$ : if then  $x^{n-1} - a^{n-1}$  be divisible by  $x - a$ , the whole of the second side of the last equation will be divisible by  $x - a$ , and therefore  $x^n - a^n$  will be divisible by  $x - a$ . If then any one of the succession

$$x - a, \quad x^2 - a^2, \quad x^3 - a^3, \quad x^4 - a^4, \quad \&c.$$

be divisible by  $x - a$ , so is the next. But this is obviously true of the first, therefore it is true of the second; being true of the second, it is true of the third; and so on, *ad infinitum*.

There are cases in which the successive induction only brings any term within the general rule, when two, three, or more of the terms immediately preceding are brought within it. Thus, in the application of this method to the deduction of the well-known consequence of

$$x + \frac{1}{x} = 2 \cos .\theta, \text{ namely, } x^n + \frac{1}{x^n} = 2 \cos .n\theta,$$

it can only be shown that any one case of this theorem is true, by showing that the preceding two cases are true: thus its truth, when  $n = 5$  and  $n = 6$ , makes it necessarily follow when  $n = 7$ . In this case the two first instances must be established (when  $n = 1$  by hypothesis, and when  $n = 2$  by independent demonstration), which two establish the third, the second and third establish the fourth, and so on.

An instance of mathematical induction occurs in every equation of differences, in every recurring series, &c.